THE GOVERNMENT OF THE REPUBLIC OF THE UNION OF MYANMAR

MINISTRY OF EDUCATION

MATHEMATICS

GRADE 10

BASIC EDUCATION CURRICULUM, SYLLABUS AND

TEXTBOOK COMMITTEE

THE GOVERNMENT OF THE REPUBLIC OF THE UNION OF MYANMAR

MINISTRY OF EDUCATION

MATHEMATICS

GRADE 10

BASIC EDUCATION CURRICULUM, SYLLABUS AND

TEXTBOOK COMMITTEE

၂၀၁၇ ခုနှစ်၊ ဒီဇင်ဘာလ၊ အုပ်ရေ (၄၂၂၂၈၂)

၂၀၁၈-၂၀၁၉ ပညာသင်နှစ်

Preface

This text is the second of two-volume set of High School Mathematics revised in the light of recent changes in international High School Mathematics curricula. The text consists of 12 chapters:

Chapter 1 introduces functional notation, composition of functions and inverse functions.

Chapter 2 is concerned with the problem of finding the remainders and factors of polynomials with the use of the Remainder Theorem and the Factor Theorem.

Chapter 3 is concerned with the Binomial Expansion of $(a + b)^n$ for the positive integral index n.

Chapter 4 is about the graphical solution of quadratic inequations.

Chapter 5 introduces the concepts of sequences and series with emphasis on Arithmetic Progression and Geometric Progression.

Chapter 6 is about matrices and basic operations : addition, multiplication and inversion. Some applications to the solution of system of linear equations are included.

Chapter 7 is concerned with the probability of an outcome. The use of tree diagrams and tables of outcomes is emphasized in calculating the probability.

Chapter 8, 9 and **10** form what might called the "geometry" part of the text. Chapter 8 is about the product properties of circles, concyclic points and converse theorems. Chapter 9 deals with relation between areas of similar triangles, while chapter 10 introduces the concept of a vector and some of its applications to geometry.

Chapter 11 starts with the definitions of six trigonometric ratios for any angle and develops some important relations (identities) between them. Two important theorems : the Law of Sines and the Law of Cosines are also included.

Chapter 12 which is the last chapter of the text is an introduction to Differential Calculus and some of its applications.

Finally, one cautionary, as well as , suggestive note on how to learn mathematics is appropriate here.

"Learn mathematics by doing only".

Do not just read your textbook. You should always have a pencil and paper with you and " work through " the text. It is suggested to make summaries of your own for each unit in each chapter and frequently review them.

CONTENTS

CHAPTER 1

Functions

We have described in the Grade 10 Text some relations and functions from one set to another set in the following ways:-

1. A verbal statement
2. An arrow diagram
3. A set of ordered pairs
4. A table form
5. A graph

For example, the function from $A = \{1, 2\}$ to $B = \{5, 10, 15\}$ described by the verbal statement "is one fifth of " may be described also in the following ways:-

Arrow diagram

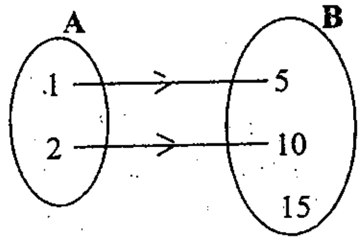


Fig. 1.1

Set of ordered pairs

$\{(1, 5), (2, 10)\}$

Table form

x	1	2
5x	5	10

Table 1.1

Graph

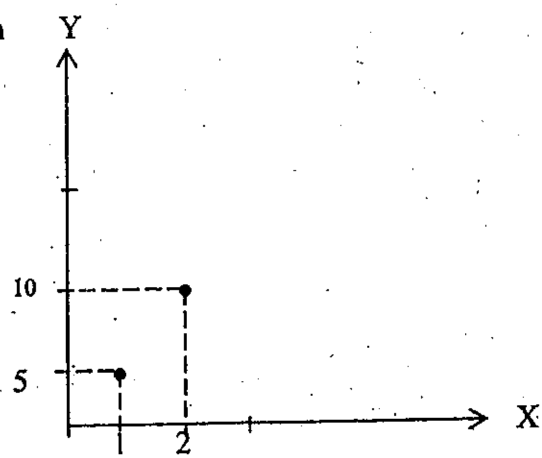


Fig. 1.2

1.1 Function

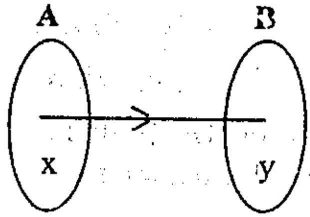


Fig. 1.3

A function from a set A to a set B relates each element of A to exactly one element of B. If x is related to y, we write

$$x \longmapsto y$$

and we say x corresponds to y.

y is called the **image** of x.

A is called the **domain** of the function.

B is called the **codomain of the function**

The set of all images is **called the** range of the function.

Thus, range of the function = $\{y \in B \mid y$ is the image of some x in A$\}$.

The range is a subset of the cod^main.

We should like to review these ideas by giving some examples:

Example 1. Let A = { 2, 3, 4} and B = { 6, 7, 9, 11, 12}. Consider a function from A to B such that $x \longmapsto 3x$ whenever $x \in A$.

Let us find the range. We have

$$2 \longmapsto 6$$
$$3 \longmapsto 9$$
$$4 \longmapsto 12$$

The range of the function is {6, 9, 12}. Here the domain is {2, 3, 4}.

The range is easily seen to be a subset of the codomain B.

Example 2. Let A = {3} and B = { 6, 7, 9, 11}. Consider a function from A to B such that $x \longmapsto 2x + 3$ where $x \in A$.

Let us find the range. We have

$$3 \longmapsto 9$$

∴ The range of the function is {9}.

Example 3. Let A = { 2, 3, 4} and B = {6, 7, 9, 11, 12}. Consider a function from A to B such that $x \longmapsto 2x + 3$ where $x \in A$.

Since we have

$$2 \longmapsto 7$$
$$3 \longmapsto 9$$
$$4 \longmapsto 11, \text{ the range is } \{7, 9, 11\}.$$

Note: If the set B does not contain the element 9, we will NOT get a function from A to B such that $x \longmapsto 2x + 3$.

From the above examples we see three things are involved in a function. They are

(i) the domain A.

(ii) the codomain B.

(iii)the rule which tells us what the image of each element of A is.

We get different functions by changing one or more of these three things in suitable ways. For example,the function in example (1) is different from the function in example (3). They differ in the "rule". **We must** name these functions in different ways. If the name of the first **function is f, we** may choose the name of the second function to be g.

Notation We write f: **A** $\longrightarrow$ **B** to denote the function from A to B **It** is read: "f is a function from A to B". If x corresponds to y, we write $f{:}x \longmapsto y$. It is read: "f maps x to y", or we say "y is the image of x under f". In example (1), we have $f: x \longmapsto 3x$, and we say that 3x is the image of x.

Example 4. Let f be a function from the set N of natural numbers to N itself such that $f: x \longmapsto 2x$ whenever $x \in N$.

That is , $f: N \longrightarrow N$ such that $f: x \longmapsto 2x$

Here the domain N = { 1, 2, 3,---}, and the codomain = N.

$f: 1 \longmapsto 2$,

$f: 2 \longmapsto 4$,

$f: 3 \longmapsto 6$, and so on.

The range = {2, 4, 6, ---}

Exercise 1.1

1. Write down the domain and range of the function f such that

$$f : 1 \longmapsto 3$$
$$f : 2 \longmapsto 5$$
$$f : 3 \longmapsto 7$$

Choose a codomain for this function.

Illustrate the function by means of an arrow diagram.

The domain is $A = \{ 0, 1, 2, 3, 4 \}$ and the function f is given by $f : x \longmapsto x + 1$ whenever $x \in A$. Find the range. Illustrate f by means of a graph.

3. Let the domain of the function $f : x \longmapsto 3x$ be the set of natural numbers less than 5.

(a) List the elements of the domain.

(b) List the elements of the range.

(c) Illustrate f by means of a graph.

4. Let the domain of the function $h : x \longmapsto 0$ be $\{ 2, 4, 6, 7\}$.

What is the range of h? Draw an arrow diagram for h.

5. Illustrate the function $f : x \longmapsto x + 2$ with an arrow diagram for the domain $\{ 3, 5, 7, 9, 10\}$. Write down the range of f.

6. Find the range of the function $g : x \longmapsto x^2$ with the domain $\{ -3, -2, -1, 0, 1, 2, 3\}$. Illustrate g by means of a graph.

7. For the function $g : x \longmapsto x^2 - 1$, copy and complete the following table:

Element of domain	−3	−2	−1	0	1	2	3
Image				−1		3	

1.2 Functional Notation

If A, B are sets, let $f : A \longrightarrow B$.

That is, f is a function from A to B.

Let $f : x \longmapsto y$

That is, y is the image of x under f.

Then y is denoted by f (x).

In other words, the image of x under f is denoted by f (x).

It is read: "f of x".

For example, let J be the set of all integers and let $f : J \longrightarrow J$ be the function such that $f : x \longmapsto x^2 + x + 1$.

In our functional notation, we write

$$f(x) = x^2 + x + 1.$$

which is read : " f of x is $x^2 + x + 1$".

It means " the image of x under f is $x^2 + x + 1$", or " the value of f(x) is $x^2 + x + 1$".

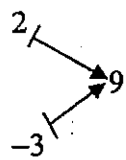
The expression $f(x) = x^2 + x + 1$ is called **the formula for the function f.**

The formula for the function is convenient way to describe the rule which tells us what the image of each element of the domain is.

Example 1. Let the function $f : R \longrightarrow R$ be given by $f(x) = x^2 + x + 3$ for every $x \in R$. Find the images of 2 and -3.

Solution: The image of 2 is $f(2) = 2^2 + 2 + 3 = 4 + 2 + 3 = 9$

The image of -3 is $f(-3) = (-3)^2 + (-3) + 3 = 9 - 3 + 3 = 9$



Note: **We find that f maps both 2 and −3 to 9.**

Therefore it is not a one-to-one function.

Example 2. Let the function $g : R \longrightarrow R$ be given by $g(x) = 3x^2$ for every $x \in R$.

Find (a) the image of 2 under g

(b) the value of $g(-1)$.

(c) the values of $a \in R$ such that $g(a) = -1$.

(d) the values of $a \in R$ such that $g(a) = 3$.

Solution

(a) $g(2) = 3(2)^2 = 3 \times 4 = 12$

(b) $g(-1) = 3(-1)^2 = 3 \times 1 = 3$

(c) $g(a) = -1$, i.e., $3a^2 = -1$

$$\therefore a = \pm\sqrt{\frac{-1}{3}}$$

Since $\pm\sqrt{\frac{-1}{3}} \notin R$, we do not have $a \in R$ such that $g(a) = -1$.

(d) $g(a) = 3$; i.e., $3a^2 = 3$

$\therefore a^2 = 1$

$\therefore a = \pm 1$

Example 3. Let the function $f: R \longrightarrow R$ be given by $f(x) = px + q$, where p and q are real numbers. If $f(1) = 4$ and $f(-2) = 1$, find p and q.

Solution $f(x) = px + q$.

$f(1) = 4$, i.e., $p + q = 4$ -----------(1)

$f(-2) = 1$, i.e., $p(-2)+q = 1$

$\therefore -2p + q = 1$ -----------(2)

Solving (1) and (2), we get $p = 1$, $q = 3$.

Example 4. Let the function $f: R \longrightarrow R$ be given by $f(x)=2^x$

(a) What are the images of 3, 0, –2?

(b) Find $a \in R$ such that $f(a) = 64$.

Solution $f(x) = 2^x$

(a) $f(3) = 2^3 = 8$

$f(0) = 2^0 = 1$

$f(-2) = 2^{-2} = \frac{1}{4}$.

(b) $f(a) = 64$

$\therefore 2^a = 64 = 2^6$

$\therefore a = 6$

Equality of functions

Two functions f and g are equal (and we write $f = g$) if and only if

(1) f and g have the same domain,

(2) f and g have the same codomain, and

(3) $f(x) = g(x)$ for each element x of the domain.

Thus, $f : A \longrightarrow B$ and $g : A \longrightarrow B$ are regarded as the same function if and only if $f(x) = g(x)$ for each x in A.

Example 5. Let $A = \{1\}$ and $B = \{1, 2\}$.

Let $f : A \longrightarrow B$ be defined by $f(x) = x^2$ and

Let $g : A \longrightarrow B$ be defined by $g(x) = 2x - 1$.

Then $f(1) = 1^2 = 1$

$g(1) = 2(1) - 1 = 1$

$\therefore f(1) = g(1)$, and so $f(x) = g(x)$ for every x in A.

$\therefore f = g$

Example 6. Let $C = \{ 1, 2 \}$ and $D = \{ 1, 3, 4 \}$.

Let $f : C \longrightarrow D$ be defined by $f(x) = x^2$, and

Let $g : C \longrightarrow D$ be defined by $g(x) = 2x-1$. Then

$f(1) = 1^2 = 1$, $g(1) = 2(1) - 1 = 1$.

$f(2) = 2^2 = 4$, $g(2) = 2(2) - 1 = 3$.

We found out that there is an element 2 in the domain C with

$f(2) \neq g(2)$.

Therefore we have $f \neq g$.

Exercise 1.2

1. Let the function $f : R \longrightarrow R$ be given by

(a) $f(x) = x + 3$. Find $f(1)$, $f(2)$, $f(-3)$, $f(0)$, $f(\frac{1}{2})$.

(b) $f(x) = 3 - 4x$. Find $f(1)$, $f(3)$, $f(-2)$, $f(\frac{1}{2})$.

(c) $f(x) = x^2 + 1$. Find $f(2)$, $f(-1)$, $f(4)$, $f(-3)$.

(d) Find $a \in R$ such that $f(a) = 50$ in (a), (b) and (c).

2. Let the function $g : R \longrightarrow R$ be given by

(a) $g(x) = 2x - 5$. Find $g(3)$, $g(\frac{1}{2})$, $g(0)$, $g(-4)$, $g(4)$. If $g(a) = 99$, find a.

(b) $g(x) = \frac{x+5}{2}$. Find the images of 3, 0, –3. Find x if $g(x) = 0$.

(c) $g(x) = 3x - 1$. Find x such that $g(x) = 20$.

(d) $g(x) = 3x + 1$. Find x such that $g(x) = 22$.

3. Let the function $f : R \longrightarrow R$ be given by

(a) $f(x) = ax + b$, where a and b are real numbers. If $f(3) = 11$ and $f(1) = 7$, find a and b.

(b) $f(x) = px + q$, where p and q are real numbers. If $f(1) = 3$ and $f(5) = 7$, find p and q.

(c) $f(x) = mx + c$, where m and c are fixed real numbers. If $f(0) = -3$ and $f(2) = 1$, find m and c, and then find $f(4)$.

4. Let the function $h : R \longrightarrow R$ be given by $h(x) = 2^x$. What are the images of 3, 2, 1, 0, –1, –2? Find $a \in R$ such that

(i) $h(a) = 64$ (ii) $h(a) = 128$ (iii) $h(a) = \frac{1}{8}$.

5. In this exercise, let f be a function from $R \longrightarrow R$.
Which of the following statements are true?

(a) If $f(x) = 5 - x$, the image of –3 under f is 8.

(b) If $f(x) = x^2 + 9$, the image of –3 under f is zero.

(c) If $f(x) = 3x + 4$, then $f(a) = a$ implies that $a = -2$.

(d) If $f(x) = x + 3$, there is only one value $a \in R$ such that $f(a) = 0$.

(e) If $f(x) = x^2 - 1$, then there are exactly two values $a \in R$ such that $f(a) = 0$.

6. If $f : R \longrightarrow R$ and $g : R \longrightarrow R$ are functions where $f(2) = 0$ and $g(2) = 0$, can we say that f and g are the same function? Why?

7. If $f : R \longrightarrow R$ is given by $f(x) = x^2 + x$, the image of any $a \in R$ can be found. If A is a subset of R, we call the set $\{ f(a) \mid a \in A \}$ the image of A under f and we denote this set by f(A). Find f(B) if $B = \{ -2, -1, 0, 1, 2 \}$.

8. Let the function $f: R \longrightarrow R$ be given by $f(x) = ax^2 + bx$. If $f(-1) = 7$ and $f(2) = -2$, find the values of a and b and then find the values of x for which $f(x) = x$.

9. The function f is defined by $f(x) = \dfrac{ax-3}{x-1}$ for all real values of x except $x = 1$.

Find the value of a for which (i) $f(2) = 5$ (ii) $f(3) = a$ (iii) $f(a) = a$.

10. The function f is defined by $f(x) = px^2 + qx - 5$ for all values of x. If $f(-1) = 1$ and $f(1) = -8$, find (a) the values of p and q.

(b) the values of x for which $f(x) = -5$.

Example 7. Let the domain of a function L be the set of positive real numbers, and let the codomain be R. Let the function be described by $L(x) = 1 + \log_{10} x$. Using your table, find, to three significant figures:

(a) the image of 12.

(b) the real number a such that $L(a) = 2.5$.

Solution

$$L(x) = 1 + \log_{10} x$$

(a) the image of 12 $= L(12) = 1 + \log_{10} 12$

$= 1 + 1.0792 = 2.0792 = 2.08$

(b) $L(a) = 1 + \log_{10} a = 2.5$

$\log_{10} a = 1.5$

$a = 31.62$

$a = 31.6$, to three significant figures.

Example 8. Let the domain of a function t be the set $\{1°, 2°, 3°, \ldots, 90°\}$ and let the codomain be R. Let the function be described by

$t: x \longmapsto \sin x + \cos x$.

(a) Use your tables to find $t(30°)$, to three significant figures.

(b) Find x such that $t(x) = 0$.

Solution

(a) $t(x) = \sin x + \cos x$

$t(30°) = \sin 30° + \cos 30° = 0.5000 + 0.8660 = 1.366 = 1.37$

(b) $t(x) = \sin x + \cos x$

Since $\sin x > 0$ and $\cos x \geq 0$ for each x in the domain,

we have, $\sin x + \cos x > 0$

Therefore there is no x in the domain such that $t(x) = 0$.

Exercise 1.3

1. Let the domain of a function L be the set of positive real numbers, and let the codomain be R. Let the function L be given by $L(x) = \log_{10}(1 + x)$. Find, to four significant figures:

   (a) the images of 5, 12 and 50. (b) "a" such that $L(a) = 2.5$.

2. Let $A = \{x \,/\, 0^\circ \leq x \leq 360^\circ\}$. **Let the function** $t : A \longrightarrow R$ be given by $t(x) = \sin x - \cos x$.

   (a) Find $t(30^\circ)$, $t(60^\circ)$, $t(90^\circ)$ to three significant figures:

   (b) If $t(\theta) = 0.6$, find θ such that $0 < \theta < 90^\circ$.

3. A function f from A to A, where A is the set of positive integers, is given by f(x) = the sum of all possible divisors of x.

   For example, $f(6) = 1 + 2 + 3 + 6 = 12$.

   (a) **Find** the values of f(2), f(5), f(13), f(18).

   (b) **Show** that $f(14) = f(15)$ and $f(3).\ f(5) = f(15)$.

4. Let A = the set of positive integers ≥ 4,

   B = the set of all positive integers.

   Let $d : A \longrightarrow B$ be a function given by $d(n) = \frac{1}{2}n(n-3)$, the number of diagonals of a polygon of n sides.

   (a) Find d(6), d(8), d(10), d(12).

   (b) How many diagonals will a polygon of 20 sides have?

1.3 Some ideas on Functions

In this section, we will study more about functions. First, we will review some ideas on functions:

Let A and B be sets. A function f from A to B is a relation in which each element of A is related to exactly one element of B. We write $f : A \longrightarrow B$.

If an element $x \in A$ is related to the element $y \in B$, we say that "y is the image of x under f" and we write

$$f(x) = y$$

we also write $f : x \longmapsto y$.

If f is a function from A to B then the set A is called the domain of the function and the set of all the images of element of A is called the range of the function.

We will write $R_f(A)$ to denote the range of f.

Thus $R_f(A) = \{ f(x) \mid x \in A \}$.

Example 1. Let $A = \{ 1, 2, 3, 4 \}$ and $B = \{ 2, 4, 6, 8, 10 \}$. Let the function $f : A \longrightarrow B$ be defined by $f(x) = 2x$. Find the range of f.

Solution

We can write down the image of each of the element of A as follows:

$f(1) = 2(1) = 2$

$f(2) = 2(2) = 4$

$f(3) = 2(3) = 6$

$f(4) = 2(4) = 8$

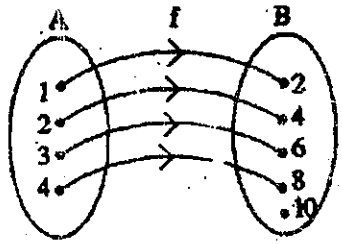


Fig 1.4

The range of the function f is

$R_f(A) = \{ 2, 4, 6, 8 \}$

In this example, the range is not exactly B itself. $R_f(A)$ is proper subset of B.

Example 2. Let $A = \{ -1, 0, 1 \}$ and $B = \{ 0, 1 \}$

Let the function $g : A \longrightarrow B$ be defined by $g(x) = x^2$. Find the range of g.

Solution

The images of elements of A are

$g(-1) = (-1)^2 = 1$

$g(0) = (0)^2 = 0$

$g(1) = (1) = 1$

The range of the function g is

$R_g(A) = \{ 0, 1 \}$ and $R_g(A) = B$.

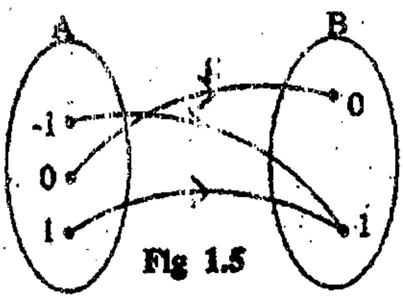


Fig 1.5

In this case, the range of g is B itself.

We notice that more than one element of A correspond to $1 \in B$.

i.e., $-1 \mapsto 1$

$1 \mapsto 1$

One - to – One correspondence

Let $f : A \longrightarrow B$ be function. If each element of B is related to exactly one element of A, then f is called a one-to-one correspondence between A and B. The sets A and B are said to be in one-to-one correspondence.

Example 3. Let A = { 1, 2, 3 } and B = { 3, 6, 9 }.

Let the function $f : A \longrightarrow B$ be defined by $f(x) = 3x$.

We have $f : 1 \mapsto 3$

$2 \mapsto 6$

$3 \mapsto 9$

We find that

$3 \in B$ is related to $1 \in A$ only

$6 \in B$ is related to $2 \in A$ only

$9 \in B$ is related to $3 \in A$ only

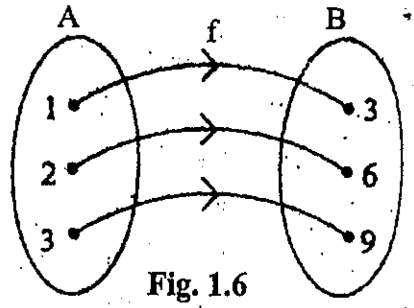


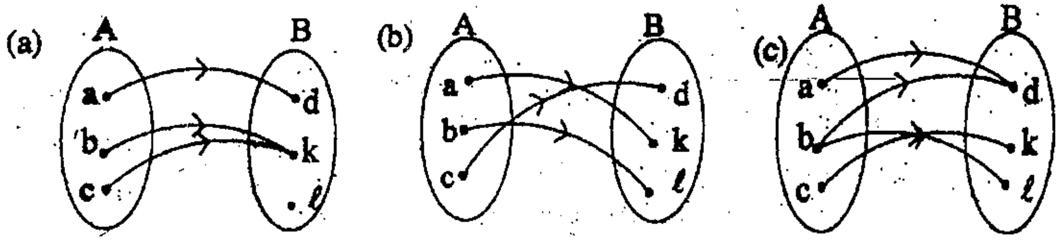
Fig. 1.6

Figure 1.6 illustrates this function.

This function is a one-to-one correspondence between A and B.

Exercise 1.4

Which of the following relations from the set A = { a, b, c }to the set B ={d, k, ℓ } is one-to-one correspondence.



The function $f : R \longrightarrow R$ is given by $f(x) = 3^x$. Which element of the domain has 243 as its image?

3. The function $g: A \longrightarrow R$, where $A = \{-2, -1, 0, 1, 2\}$, is given by $g(x) = x^2 + 1$. Find the range of g.

4. The function $h: R \longrightarrow R$ is defined by $h(x) = x^2$. Find the images of the elements $-3, -2, -1, 0, 1, 2, 3$. State the range of h.

Some useful functions

(1) Constant function

Let $f: R \longrightarrow R$ be given by $f(x) = k$, where $k \in R$ is a constant. That is k is fixed and $f(x) = k$ for every $x \in R$. Each real number $x \in R$ is related to k. Thus

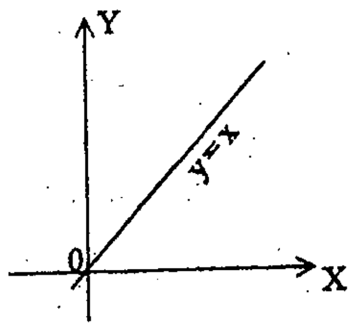
$$f(-1) = k$$
$$f(0) = k$$
$$f(4) = k$$
$$f(5) = k, \text{---etc.}$$

The function f is called a constant function.

(2) The identity function on A

Let A be any set. The function $I: A \longrightarrow A$ defined by $I(x) = x$ is called the identity function on A. Each element of A is related to itself.

I is a one-to-one correspondence between A and A.



(3) The modulus function

If x is a real number, we define

$$|x| = x, \quad \text{if } x \geq 0$$

and

$$|x| = -x, \quad \text{if } x < 0.$$

$|x|$ is called the modulus, or the absolute value of x.

For example, $|2| = 2$, $|-5| = 5$, $|0| = 0$.

The function $f: R \longrightarrow R$ defined by $f(x) = |x|$ is known as the modulus function.

Example Sketch the graph of the modulus function, $y = |x|$.

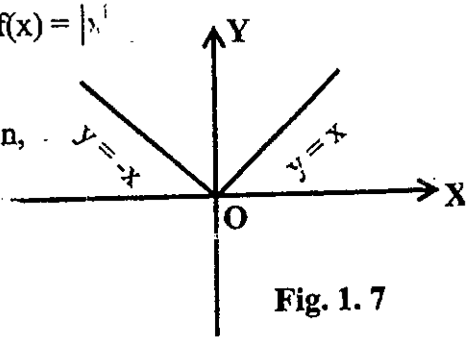


Fig. 1.7

(4) Step function

Let $A = \{x \mid 0 \leq x \leq 3\}$ and $B = R$. Let $f : A \longrightarrow B$ be defined by

$$f(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ 1 & \text{when } 1 \leq x < 2 \\ 2 & \text{when } 2 \leq x < 3 \end{cases}$$

This function is called a **step function.**

The graph is

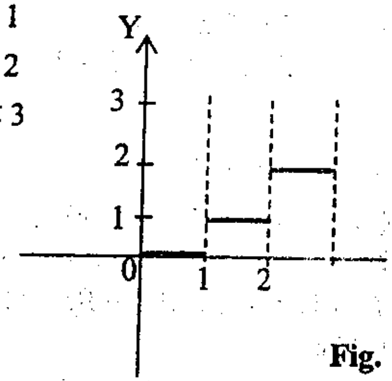


Fig.

1.4 Composition of Functions

We will start with an illustration.

Let $f : R \longrightarrow R$ be defined by $f(x) = 3x$ and

Let $g : R \longrightarrow R$ be defined by $g(x) = x - 5$.

Consider $2 \in R$.

The image of 2 under f is $f(2) = 3(2) = 6$.

The image of 6 under g is $g(6) = 6 - 5 = 1$

We may draw a flow chart as follows:

$$2 \xmapsto{f} 6 \xmapsto{g} 1$$

We used f first, and then g.

If x is an element of R, we have

$$x \xmapsto{f} 3x \xmapsto{g} 3x - 5$$

Thus $f(x) = 3x$, and $g(3x) = 3x - 5$

We get a function from R to R by taking $3x - 5$ as the image of x for each $x \in R$.

This function is denoted by $g \circ f$. Thus $g \circ f : R \longrightarrow R$, where

$$(g \circ f)(x) = 3x - 5.$$

$g \circ f$ is called the **composite of f and g**, and is read " g circle f".

Formula of the composite

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be given functions.

Note that the range of f is a subset of the domain of g.

Let $x \in A$ be any element. Using f first and then g we have,

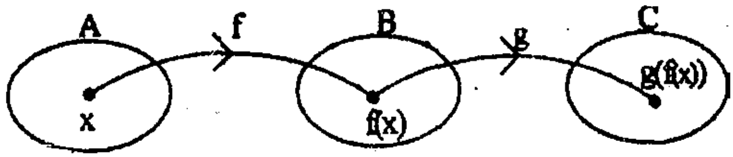


Fig. 1.9

$x \stackrel{f}{\longmapsto} f(x) \stackrel{g}{\longmapsto} g(f(x))$. Thus $g \circ f : A \longrightarrow C$, where

$(g \circ f)(x) = g(f(x))$.

Example 4. Function $f : R \longrightarrow R$ and $g : R \longrightarrow R$ are defined by $f(x) = x^2$ and $g(x) = 3x + 1$. Find (i) $(g \circ f)(2)$ (ii) $(f \circ g)(2)$

(iii) the formulae of $g \circ f$ and $f \circ g$.

Solution

(i) $(g \circ f)(2) = g(f(2)) = g(4) = 12 + 1 = 13$

(ii) $(f \circ g)(2) = f(g(2)) = f(7) = 7^2 = 49$

(iii) $g \circ f : R \longrightarrow R$ where

$(g \circ f)(x) = g(f(x)) = g(x^2) = 3x^2 + 1$

$f \circ g : R \longrightarrow R$, where

$(f \circ g)(x) = f(g(x)) = f(3x + 1) = (3x + 1)^2$

Exercise 1.5

1. Functions $f : R \longrightarrow R$ and $g : R \longrightarrow R$ are defined by $f(x) = 2x + 1$ and $g(x) = 3x$.
   (a) Calculate $(g \circ f)(1)$ and $(g \circ f)(3)$.
   (b) Find the formula of $g \circ f$ and check the above images.
2. The mappings $f : R \longrightarrow R$ and $g : R \longrightarrow R$ are defined by $f(x) = x + 2$ and $g(x) = x^2$. Find (a) $(g \circ f)(-1)$, $(g \circ f)(2)$ and $(g \circ f)(x)$.
   (b) $(f \circ g)(-1)$, $(f \circ g)(2)$ and $(f \circ g)(x)$.
3. Repeat question 2 for the mappings $f : R \longrightarrow R$ and $g : R \longrightarrow R$ defined by $f(x) = x + 1$ and $g(x) = x^3$.
4. For the functions $g : R \longrightarrow R$ and $h : R \longrightarrow R$ defined by $g(x) = 2x$ and $h(x) = x^2 + 4$, find, in simplest form:
   (a) $(h \circ g)(x)$ (b) $(g \circ h)(x)$ (c) $(g \circ g)(x)$ (d) $(h \circ h)(x)$.

5. All of the followings are functions from $R \longrightarrow R$. Find a formula for $g \circ f$ in each case.

(a) $f(x) = x - 1$, $g(x) = x^2$.

(b) $f(x) = x + 1$, $g(x) = 2x^2 - x + 3$.

(c) $f(x) = x^2 - 1$, $g(x) = 3x + 1$.

(d) $f(x) = -x$, $g(x) = x$.

(e) $f(x) = x^2$, $g(x) = \dfrac{1}{x^2 + 2}$.

(f) $f(x) = 2x - 3$, $g(x) = x^2 + 5$.

(g) $f(x) = 5x^2$, $g(x) = \dfrac{1}{x^2 + 1}$.

6. A function $f : R \longrightarrow R$ is defined by $f(x) = x + 1$. Find the function $g : R \longrightarrow R$ in each of the following:

(a) $(g \circ f)(x) = x^2 + 5x + 5$. (b) $(f \circ g)(x) = x^2 + 5x + 5$.

7. If $f : R \longrightarrow R$ is defined by $f(x) = x^2 + 1$, find the function g such that $(f \circ g)(x) = x^2 + 4x + 5$.

8. If $f : R \longrightarrow R$ is defined by $f(x) = x^2 + 3$, find the function g such that $(g \circ f)(x) = 2x^2 + 3$.

9. Functions f and g are defined by $f(x) = px - 2$, where p is a constant, and $g(x) = 4x + 3$. Find (a) an expression for $(f \circ g)(x)$.

(b) the value of p for which $(f \circ g)(x) = (g \circ f)(x)$.

10. If $f : R \longrightarrow R$ and $g : R \longrightarrow R$ is defined by $f(x) = ax + b$, where a and b are constants, $g(x) = x + 7$, $(f \circ g)(1) = 7$ and $(f \circ g)(2) = 15$, find $(f \circ g)(3)$.

1.5 Some Properties of Composition of Functions

Composition of functions is an algebraic operation in the set of functions. It is very useful in investigating the algebraic properties of functions.

In this section we will illustrate some properties of composition of functions by working out particular examples.

Closure

By the definition of composition of two functions, the composite of functions is again a function. In most applications we work on particular sets of functions, and we should like the composite of two functions of a certain type to be of the same type.

Example 1. Let $f : R \longrightarrow R$ and $g : R \longrightarrow R$ be defined by $f(x) = 2x + 3$ and $g(x) = 5x - 4$. (Functions of this type are known as linear functions).

Let us compute $f \circ g$ and $g \circ f$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(5x - 4) \\ &= 2(5x - 4) + 3 \\ &= 10x - 5\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(2x + 3) \\ &= 5(2x + 3) - 4 \\ &= 10x + 11\end{aligned}$$

We see that both $f \circ g$ and $g \circ f$ are linear functions.

We say that "linear functions are closed under composition". The above example provides a particular verification.

Example 2. Let $A = \{1, 2, 3\}$. Let the function $f : A \longrightarrow A$ and $g : A \longrightarrow A$ be defined by the arrow diagrams.

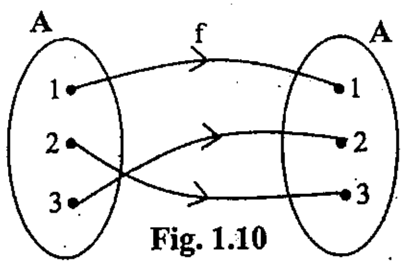


Fig. 1.10

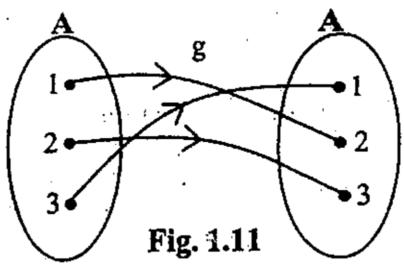


Fig. 1.11

Note that f and g are one-to-one correspondences.

We may compute the composites $f \circ g$ and $g \circ f$ by means of arrow diagrams:

The composite $f \circ g$

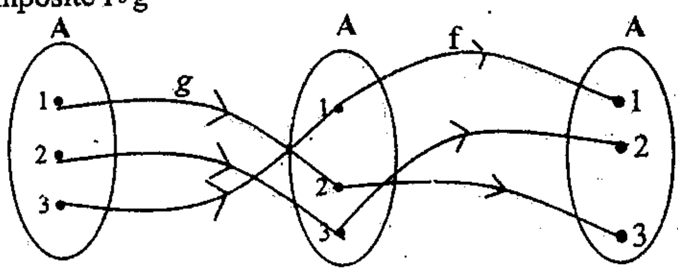


Fig. 1.12

$$\text{Thus } (f\circ g)(1) = 3$$
$$(f\circ g)(2) = 2$$
$$(f\circ g)(3) = 1$$

We observe that $f\circ g : A \longrightarrow A$ is again a one-to-one correspondence. This example illustrate the property:

One-to-one correspondence between A and A are closed under composition. We leave the verification for $g\circ f$ as an exercise.

Associative Property

Example 3. $f : R \longrightarrow R$, $g : R \longrightarrow R$ and $h : R \longrightarrow R$ are functions defined by $f(x) = 3x$, $g(x) = x - 1$, $h(x) = x^2$.

Let us compute the image of 2 under the composites $h\circ(g\circ f)$ and $(h\circ g)\circ f$

$$(h\circ(g\circ f))(2) = h((g\circ f)(2)) = h(g(f(2))) = h(g(6)) = h(5) = 25$$
$$((h\circ g)\circ f)(2) = (h\circ g)(f(2)) = (h\circ g)(6) = h(g(6)) = h(5) = 25$$

Thus $(h\circ(g\circ f))(2) = ((h\circ g)\circ f)(2)$

We have not proved that $h\circ(g\circ f) = (h\circ g)\circ f$.

The above working provides just a particular verification. It illustrates the associative property of the composition of functions.

Exercise. Repeat the above verification for the image of (–5). Show that the formula for $h\circ(g\circ f)$ in example (3) is $(h\circ(g\circ f))(x) = (3x - 1)^2$. Compute the images of 2 and –5 by using this formula.

Identity function

Example 4. The identity function I on R is the function $I : R \longrightarrow R$ defined by $I(x) = x$.

That is, the image of every $x \in R$ is just itself.

Let $f : R \longrightarrow R$ be defined by $f(x) = 3x + 2$. Then

$$(f\circ I)(x) = f(I(x)) = f(x)$$
$$(I\circ f)(x) = I(f(x)) = I(3x + 2) = 3x + 2 = f(x)$$

Thus $(f\circ I)(x) = f(x)$ and

$(I\circ f)(x) = f(x)$ for every $x \in R$.

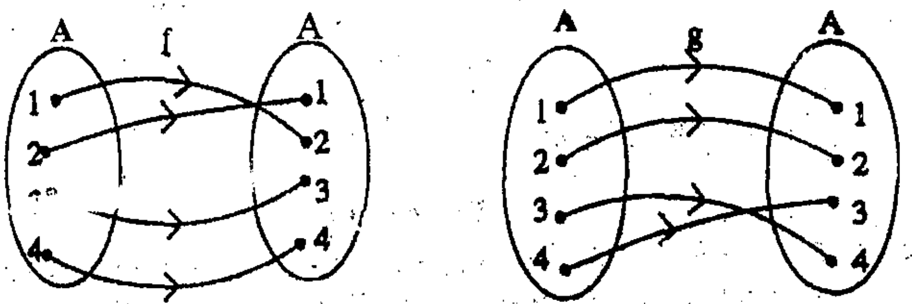
We say that $f \circ I = f$ and $I \circ f = f$

Exercise. Let $g : R \longrightarrow R$ be defined by $g(x) = x^2 - 3$ and let $I : R \longrightarrow R$ be the identity function. Show that $g \circ I = g$ and $I \circ g = g$.

Commutativity

The composition of functions **does not**, in general, obey the commutative law.

Example 5. Let $A = \{ 1, 2, 3, 4 \}$. Let the function $f : A \longrightarrow A$ and $g : A \longrightarrow A$ be defined by the arrow diagrams:



(i) (ii)

Fig. 1.13

We can check that, in this particular case $f \circ g = g \circ f$.

These two particular functions commute in the composition. But $f \circ g$ and $g \circ f$ may be unequal in other cases.

Example 6. Consider the functions f and g of example 1.

We have $(f \circ g)(x) = 10x - 5$.

$\therefore (f \circ g)(1) = 10 - 5 = 5$

But $(g \circ f)(x) = 10x + 11$

$\therefore (g \circ f)(1) = 10 + 11 = 21$

Hence $f \circ g \neq g \circ f$

Example 7. Let A = { 1, 2, 3, 4}. Let the functions $f : A \longrightarrow A$ and $g : A \longrightarrow A$ be defined by the arrow diagrams:

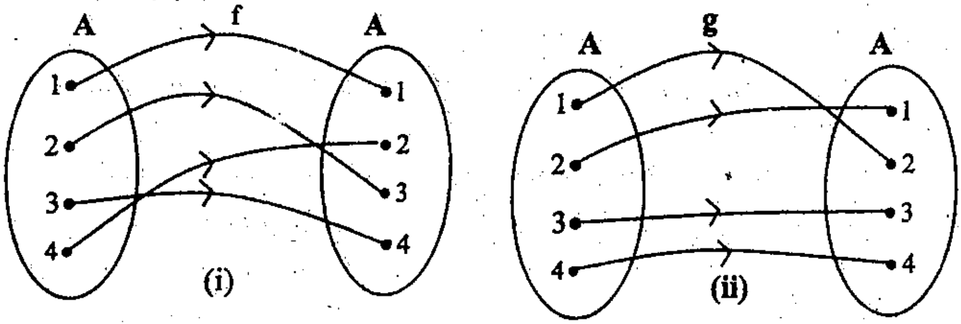


Fig. 1.14

We can check that $f \circ g \neq g \circ f$.

Exercise 1.6

1. $f : R \longrightarrow R$, $g : R \longrightarrow R$ and $h : R \longrightarrow R$ are functions defined by $f(x) = x - 2$, $g(x) = x^3$ and $h(x) = 4x$. Show that $((h \circ g) \circ f)(x) = 4(x-2)^3$ and $((f \circ g) \circ h)(x) = 64x^3 - 2$. Calculate $((h \circ g) \circ f)(1)$ and $((f \circ g) \circ h)(1)$.
2. Let I be the identity function on R and let $f : R \longrightarrow R$ be defined by $f(x) = x^2 + x + 3$. Show that $I \circ f = f \circ I = f$.
3. Compute $(f \circ g)(1)$ and $(g \circ f)(1)$ in example 7.

1.6 Inverse Functions

We will introduce the idea of the "inverse" of a function by given some simple examples.

Example 1. Let A = { 1, 2 }, B = { 3, 6}. Let the function $f : A \longrightarrow B$ be defined by the arrow diagram:

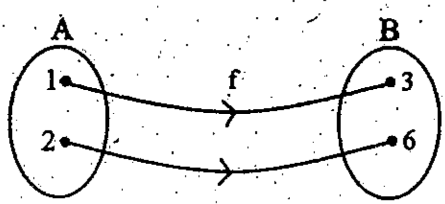


Fig. 1.15 (i)

Now let us reverse the direction of the arrow. We get the diagram.

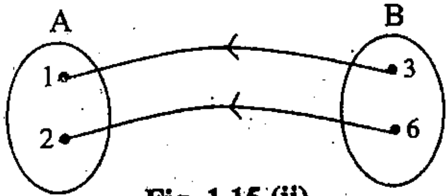


Fig. 1.15 (ii)

Our main purpose is to obtain a function from B to A.

At least in this case we do get what we want. The function we obtained is, in general, not the same as the given function. (We may call it the inverse of f and it is denoted by f^{-1}.) f^{-1} is read : " f inverse" or "inverse of f".

We should notice that

$f(1) = 3$ whereas $f^{-1}(3) = 1$

$f(2) = 6$ whereas $f^{-1}(6) = 2$

Most important of all is to observe that this function f happens to be a one-to- one correspondence between the two given sets A and B. The condition "each element of B is related to exactly one element of A" ensures the existence of the inverse function.

Unfortunately, some cases are not favourable. Let us look at a couple of functions in the following examples.

Example 2. Let the function $f : A \longrightarrow B$ be given by the arrow diagram. Note that the given function is not a "one-to-one correspondence".

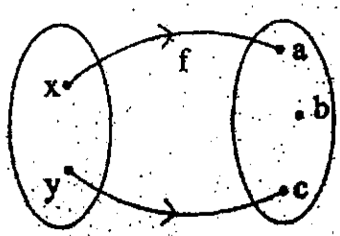


Fig. 1.16 (i)

If we reverse the direction of the arrows, we get the diagram:

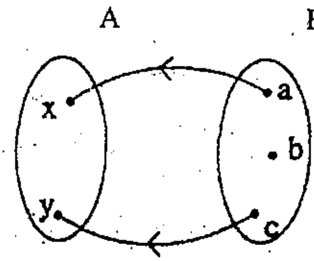


Fig. 1.16 (ii)

This does not give us a function from B to A. We do not obtain the image of b $\in$ B among the reversed arrows. We say that the inverse of the given function f does not exist.

We observe that when the function f is not a one-to-one correspondence, the inverse of the given function f does not exist, i.e., f^{-1} does not exist.

Example 3. Let the function $f : A \longrightarrow B$ be given by the arrow diagram:

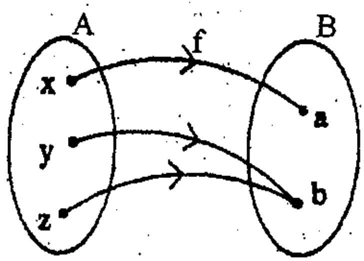


Fig.1.17(i)

If we reverse the direction of the arrows, we get the diagram:

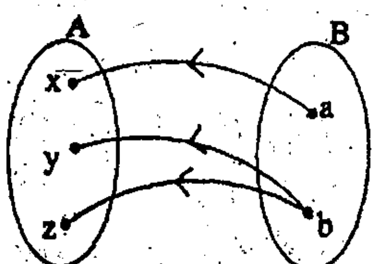


Fig.1.17 (ii)

The element b $\in$ B is related to more than one element of A. This does not give us a function from B to A.

We do not get an inverse of the given function.

Observe again that the given function f is not a one-to-one correspondence, and that f^{-1} does not exist.

Condition for Existence

[If we have a function $g : B \longrightarrow A$ such that $g(b) = x$ whenever $f(x) = b$, then g is called the inverse of f, and we write $g = f^{-1}$.]

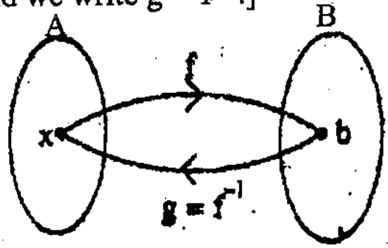


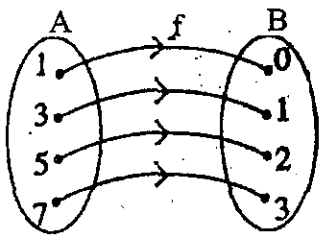
Fig. 1.18

If the function g exists, then inverse of g is f. We say that f and g are inverse functions, i.e., each is the inverse of the other.

A function $f : A \longrightarrow B$ has the inverse function $g : B \longrightarrow A$ if and only if f is a one-to-one correspondence between A and B. We write f^{-1} for the function g.

Exercise 1.7

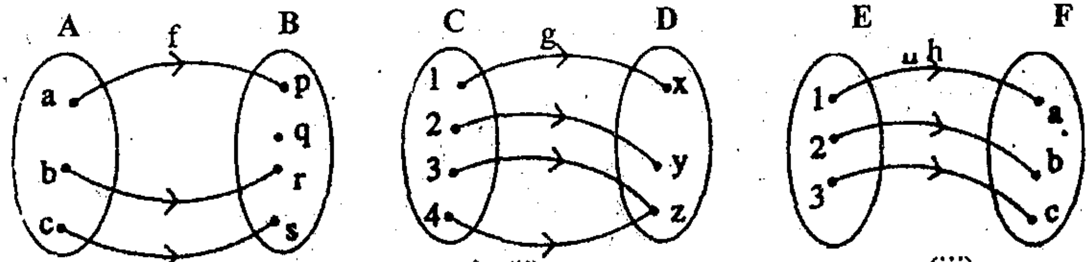
1. $f : A \longrightarrow B$ is defined by the arrow diagram:



Show in an arrow diagram the inverse of f.

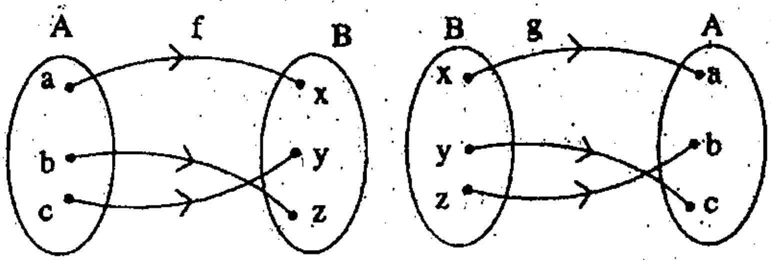
2. Let A = { 1, 2, 3, 4 } and B = { 4, 5, 6, 7 }.Let $f : A \longrightarrow B$ be defined by $f(x) = x + 3$.
   (a) Show f in an arrow diagram.
   (b) Show f^{-1} in another arrow diagram.
3. Let A ={– 2, –1, 0, 1, 2} and B ={0,1,4}.Let $f : A \longrightarrow B$ be defined by $f(x) = x^2$.
   (a) Show f in an arrow diagram.
   (b) Explain why f^{-1} does not exist.

4. The following arrow diagrams represent the functions $f : A \longrightarrow B$, $g : C \longrightarrow D$ and $h : E \longrightarrow F$ respectively. Decide in which case reversing the arrows gives a function from the second set to the first set.



(i) (ii) (iii)

5. $f : A \longrightarrow B$ and the inverse function $g : B \longrightarrow A$ are given by the diagram:



(a) Find $(g \circ f)(a)$, $(g \circ f)(b)$ and $(g \circ f)(c)$.

(b) Find $(f \circ g)(x)$, $(f \circ g)(y)$ and $(f \circ g)(z)$.

(c) Which of the following functions are equal?

f, g, $I_A : A \longrightarrow A$ and $I_B : B \longrightarrow B$.

1.7 Finding a Formula for Inverse Function

Let the function $f : A \longrightarrow B$ be given by the arrow diagram:

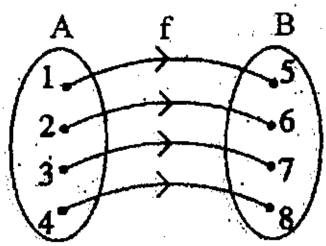


Fig. 1.19 (i)

If we reverse the direction of the arrows, we get the diagram

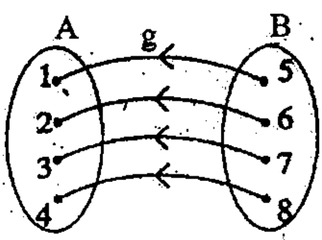
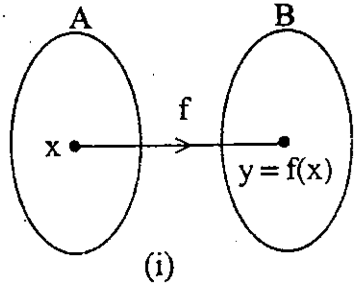
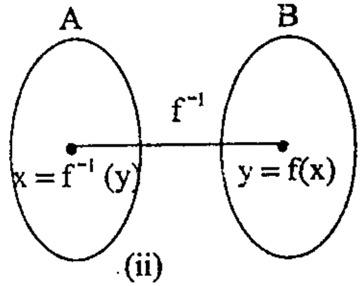


Fig. 1. 19(ii)

A new function g having domain B and range A is formed from the function f. The new function g is called the inverse of f and denoted by f^{-1}.

(Note : f^{-1} exists as a function only when f is a one-to-one correspondence). Fig. 1.20 shows the generalised process of obtaining the inverse function.





From Fig. 1.20, we have, in general

$$y = f(x) \Leftrightarrow x = f^{1}(y)$$

Example 1. Let $f : R \longrightarrow R$ be defined by $f(x) = 2x + 5$ and let y be the image of x under f. Find the formula for f^{-1}.

$$f(x) = y$$

$$\therefore 2x + 5 = y$$

$$2x = y - 5$$

$$x = \frac{y-5}{2}$$

$$f^{-1}(y) = \frac{y-5}{2}$$

Thus, the inverse function is, $f^{-1}(x) = \dfrac{x-5}{2}$

Example 2. Find the formula for f^{-1}, the inverse function of f defined by $f(x) = \dfrac{2}{3-4x}$. State the suitable domain of f.

Let y be the image of x under f.

$$f(x) = y$$

$$\therefore \frac{2}{3-4x} = y$$

$$2 = y(3-4x) = 3y - 4yx$$

$$(4y)x = 3y - 2$$

$$x = \frac{3y-2}{4y}$$

Thus, the required formula is $f^{-1}(x) = \dfrac{3x-2}{4x}, x \neq 0.$

In $f(x) = \dfrac{2}{3-4x}$, we must have $3 - 4x \neq 0.$

$$\therefore x \neq \frac{3}{4}$$

$\therefore$ Domain of f is $A = \{x \mid x \in R, x \neq \frac{3}{4}\}$

Example 3. A function $f: R \longrightarrow R$ be defined by $f(x) = \dfrac{2x+3}{x-5}$, $x \neq 5$, find $f^{-1}(3)$.

Let y be the image of x under f

$$f(x) = y \Leftrightarrow x = f^{-1}(y)$$

Let $a = f^{-1}(3)$

$\therefore a = f^{-1}(3) \Leftrightarrow f(a) = 3$

i.e. $\dfrac{2a+3}{a-5} = 3$

$$2a + 3 = 3a - 15$$

$$a = 18$$

$$\therefore f^{-1}(3) = 18$$

Exercise 1.8

1. Find the formula for the inverse function f^{-1} where $f: R \longrightarrow R$ is defined by

(a) $f(x) = 2x - 3$ (b) $f(x) = 1 + 3x$ (c) $f(x) = 1 - x$

(d) $f(x) = \dfrac{x+9}{2}$ (e) $f(x) = \dfrac{1}{3}(4x - 5)$ (f) $f(x) = \dfrac{2x+5}{x-7}, x \neq 7.$

2. $A = \{ x | x > 0, x \in R \}$ and f, g, h are functions from R to R defined by $f(x) = x + 1, g(x) = 2x, h(x) = x^2.$

(a) Find the formulae for the inverse functions f^{-1}, g^{-1}, h^{-1}.

(b) Evaluate $f^{-1}(5)$, $g^{-1}(7)$, and $h^{-1}(5)$.

3. Functions $f: R \longrightarrow R$ and $g: R \longrightarrow R$ are defined by $f(x) = 2x$ and $g(x) = x + 2$.

(a) Find formulae for the inverse functions f^{-1} and g^{-1}.

(b) Find formulae for $g \circ f$, $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$.

4. A function f is defined by $f(x) = \dfrac{2x - 5}{x - 3}$.

(a) State the value of x for which f is not defined.

(b) Find the value of x for which $f(x) = 0$.

(c) Find the inverse function f^{-1}, and state the domain of f^{-1}.

5. Find the formulae for f^{-1} where the function f is defined by

(a) $f(x) = \dfrac{1}{1+x}, x \neq -1$ (b) $f(x) = \dfrac{3}{x-2}, x \neq 2$

(c) $f(x) = \dfrac{x}{x-4}, x \neq 4$

(d) $f(x) = \dfrac{3x-5}{2x+7}, x \neq \dfrac{7}{2}$ (e) $f(x) = \dfrac{13}{2x}, x \neq 0$

6. Let $f: R \longrightarrow R$ and $g: R \longrightarrow R$ be defined by $f(x) = 3x - 1$ and $g(x) = x+7$. Find $(f^{-1} \circ g)(x)$ and $(g^{-1} \circ f)(x)$. What are the values of $(f^{-1} \circ g)(3)$ and $(g^{-1} \circ f)(2)$?

7. A function f is defined by $f(x) = 3x - 1$.

(a) Find $(f \circ f)(x)$ and $f^{-1}(x)$.

(b) Determine whether $(f \circ f)^{-1}(x)$ is the same as $(f^{-1} \circ f^{-1})(x)$.

8. A function f is defined by $f(x) = \dfrac{x+10}{x-8}, x \neq 8$. Find

(a) $f^{-1}(5)$

(b) a positive number p such that $f(p) = p$.

9. The functions f and g are defined for real x as follows:

$$f(x) = 2x - 1 \text{ and } g(x) = \frac{2x+3}{x-1}, x \neq 1.$$

(a) Find the composite functions $f \circ g$ and $g \circ f$.

(b) Find the inverse functions f^{-1} and g^{-1}.

(c) Evaluate $(f \circ g^{-1})(1)$ and $(g^{-1} \circ f^{-1})(2)$.

10. Given that $f(x) = \frac{x+a}{x-2}$, $x \neq 2$ and that $f(7) = 2$, find

(a) the value of a.

(b) $f^{-1}(-4)$.

1.8 Binary Operation

The reader is already familiar with several operations such as addition, subtraction, multiplication and division in the context of a set of real numbers. For example, we can combine the natural numbers 2 and 3 by means of addition to obtain the natural number 5. In fact, we are associating the ordered pair (2,3) with the natural number 5.

Under this rule

the ordered pair (3, 1) will give $3 + 1 = 4$

the ordered pair (4, −2) will give $4 + (-2) = 2$ etc.

In functional view point, we are looking at the function

$$f : N \times N \longrightarrow N$$
$$(x,y) \longmapsto f((x,y)) = x + y$$

where N is the set of Natural numbers. Let us be more specific and state the following definition.

Definition 1: A binary operation "Θ" on a set A is a function from A x A into A. The domain of "Θ" is A x A and the range of "Θ" is a subset of A.

The property that the range of the operation is a subset of A is referred to as **closure** of the operation. In other words, a binary operation must be closed in such a way that the image of the ordered pair (x, y), which we denote by $x \,\Theta\, y$ must be in A, i.e., $\Theta(x,y) = x \,\Theta\, y \in A$ whenever $(x,y) \in A \times A$. If the operation Θ maps (x, y) of

A x A into t $\in$ A, then we shall denote this by $\odot(x, y) = x \odot y = t$. In short, we write $x \odot y = t$.

Remark : (1) If N is the set of natural numbers, then the function

$$\odot: N \times N \longrightarrow N \text{ defined by}$$
$$(x,y) \longmapsto x \odot y = x + y$$

is a binary operation. That is, addition is a binary operation on the : of natural numbers.

(2) Similarly, multiplication

$$\odot: N \times N \longrightarrow N$$
$$(x,y) \longmapsto x \odot y = xy \text{ is a binary operation.}$$

Example 1. Let the mapping $\odot$ be defined by $(x, y) \longmapsto x \odot y = x + 2y$, where $x, y \in A = \{0, 1\}$. Is this function a binary operation?

Solution The domain of the function $\odot$ is A x A. We have to find out whether the range of the function is a subset of A, so that the function will satisfy the closure property. First, note that, as $A = \{0, 1\}$

$$A \times A = \{(0,0), (0,1), (1,1), (1,0)\}$$

Under the mapping

$$(0,0) \longmapsto 0 \odot 0 = 0 + 2.0 = 0$$
$$(0,1) \longmapsto 0 \odot 1 = 0 + 2.1 = 2$$
$$(1,0) \longmapsto 1 \odot 0 = 1 + 2.0 = 1$$
$$(1,1) \longmapsto 1 \odot 1 = 1 + 2.1 = 3$$

Thus the range of $\odot$ is $\{0, 1, 2, 3\}$, which is not a subset of A. Hence the closure property is NOT satisfied. Thus the mapping $\odot$ is not a binary operation on A. (Actually, it is just sufficient to point out that the image of (0, 1), which is 2, is not in A.)

Remark : The simplest way to show the elements produced by a binary operation is by construction of a table (known as Cayley table) as shown in Fig. 1.21 below. The element $a \odot b$ can be found at the intersection of the row containing a and the column containing b.

Θ	a	b	c
a		a Θ b	
b			
c			
⋮			

row containing element a

Fig. 1.21

column containing element b.

For example, Cayley table for the above example 1 is shown in Fig. 1.22.

Θ	0	1	
0	0	2	
1	1	3	

Θ is defined by x Θ y = $x + 2y$.

Fig. 1.22

Construction of Cayley table can be seen in example 6.

Example 2. Let N be the set of natural number. Is the function Θ defined by a Θ b = a (a + b) where a, b ∈ N, a binary operation? If it is a binary operation, find (a) 2 Θ 3, (b) 2 Θ 2, (c) 3 Θ 2, (d) Is 3 Θ 2 = 2 Θ 3?

Solution

Here the domain of the function Θ is N × N.

Θ (a,b) = a Θ b = a (a + b).

We must show that the closure property is satisfied.

That is, to show that image Θ (a, b) = a Θ b ∈ N.

Since a, b are natural numbers, their sum (a + b) and the product a (a + b) are also natural numbers.

Hence a (a + b) ∈ N.

Thus a Θ b ∈ N and therefore the closure property is satisfied. The function Θ is a binary operation.

Now a Θ b= a (a + b), where a, b∈ N.

$\therefore$ (a) $2 \odot 3 = 2\,(2+3) = 10$

(b) $2 \odot 2 = 2\,(2+2) = 8$

(c) $3 \odot 2 = 3\,(3+2) = 15$

(d) $2 \odot 3 \neq 3 \odot 2$ (by (a) and (c)).

The property of the above example means that the binary operation is not commutative. In general we can define commutative ness as follows:

Definition 2: A binary operation $\odot : A \times A \longrightarrow A$

$$(a, b) \longmapsto a \odot b$$

is said to be commutative if and only if,

$$a \odot b = b \odot a.$$

(Here ordering of the ordered pair is very important.)

Example 3. A binary operation on the set R of real numbers is defined by $x \odot y = xy + x + y$. Show that $x \odot y = y \odot x$. (i.e, to show that binary operation is commutative.)

Solution

$x \odot y = xy + x + y$

$y \odot x = yx + y + x$

Clearly $x \odot y = y \odot x$.

(Thus the binary operation is commutative)

We have seen composition of two functions f and g. For example composition $g \circ f$ is given by $(g \circ f)(x) = g(f(x))$.

Since a binary operation is a function, we can composite it twice. Thus if $\odot$ is a binary operation

$$\odot : A \times A \longrightarrow A$$
$$(a,b) \longmapsto a \odot b.$$

then $(a \odot b) \odot c$ means, first we have to take the operation $(a,b) \longmapsto a \odot b \in A$, and again to take the operation $(a \odot b, c) \longmapsto (a \odot b) \odot c$.

Example 4. Let R be the set of real numbers. A binary operation $\odot$ is defined by

$$\odot : R \times R \longrightarrow R$$
$$(x, y) \longmapsto x \odot y = x^2 + y^2$$

(a) Evaluate $[(2 \odot 3) \odot 4] + [2 \odot (3 \odot 4)]$.

(b) Show that $(x \odot y) \odot x = x \odot (y \odot x)$.

Solution

(a) $\odot: R \times R \longrightarrow R$

$(x,y) \longmapsto x \odot y = x^2 + y^2$

$\therefore \quad 2 \odot 3 = 2^2 + 3^2 = 4 + 9 = 13$

$(2 \odot 3) \odot 4 = 13 \odot 4$

$= 13^2 + 4^2 = 169 + 16 = 185.$ ------------- (1)

Next $\quad 3 \odot 4 = 3^2 + 4^2 = 9 + 16 = 25$

$\therefore \quad 2 \odot (3 \odot 4) = 2 \odot 25$

$= 2^2 + 25^2 = 4 + 625 = 629$ ------------- (2)

$\therefore \quad (2 \odot 3) \odot 4 + 2 \odot (3 \odot 4) = 185 + 629 = 814$

(b) To find $(x \odot y) \odot x$, we have to evaluate first $x \odot y$. Now $x \odot y = x^2 + y^2$

$\therefore \quad (x \odot y) \odot x = (x^2 + y^2) \odot x = (x^2 + y^2)^2 + x^2$

$= x^4 + 2x^2y^2 + y^4 + x^2$ ------------- (3)

On the other hand,

$y \odot x = y^2 + x^2.$

$\therefore \quad x \odot (y \odot x) = x \odot (y^2 + x^2) = x^2 + (y^2 + x^2)^2$

$= x^2 + y^4 + 2x^2y^2 + x^4$ ------------- (4)

From (3) and (4), we conclude that $(x \odot y) \odot x = x \odot (y \odot x)$.

Note: From the equations (1) and (2) of the above example, we can see that

$(2 \odot 3) \odot 4 = 185$

and $\quad 2 \odot (3 \odot 4) = 629.$

thus $\quad (2 \odot 3) \odot 4 \neq 2 \odot (3 \odot 4).$

In this case, we say that the binary operation is not **associative.** In general we can define associativeness as follows:

Definition 3: A binary operation $\odot: A \times A \longrightarrow A$

$(x, y) \longmapsto x \odot y$

is said to be associative if

$a \odot (b \odot c) = (a \odot b) \odot c$, for any $a, b, c \in A$.

Example 5. A binary operation Θ on the set R of real numbers is defined by $x \Theta y = xy + x + y$. Show that $(x \Theta y) \Theta z = x \Theta (y \Theta z)$.

Solution

To find $(x \Theta y) \Theta z$, we first find $(x \Theta y)$.

$$x \Theta y = xy + x + y = t, \text{ (say)}$$

$$\therefore \quad (x \Theta y) \Theta z = t \Theta z = tz + t + z.$$

Substituting $t = xy + x + y$, we get

$$(x \Theta y) \Theta z = (xy + x + y)z + (xy + x + y) + z$$

$$= xyz + xz + yz + xy + x + y + z \text{ ---------- (1)}$$

On the other hand, to find $x \Theta (y \Theta z)$, we first find $y \Theta z$.

$$y \Theta z = yz + y + z = s, \text{ (say)}$$

$$\therefore \quad x \Theta (y \Theta z) = x \Theta s$$

$$= xs + x + s$$

Substituting $s = yz + y + z$, we get

$$x \Theta (y \Theta z) = x(yz + y + z) + x + (yz + y + z)$$

$$= xyz + xy + xz + x + yz + y + z$$

$$= xyz + xy + xz + yz + x + y + z \quad \text{---------- (2)}$$

By (1) and (2), we get

$$(x \Theta y) \Theta z = x \Theta (y \Theta z).$$

Example 6. A binary operation Θ on the set of integers defined by $x \Theta y$ = the remainder when x^y is divided by 5. Complete the following operation table. (Cayley table).

[Here, 0^0 is indeterminate, so we assume $0^0 = 1$]

Θ	0	1	2	3	4
0	1				
1		1			
2			4		
3				2	
4					1

Table 1.2

Solution $x \odot y$ = the remainder when x^y is divided by 5.

$0 \odot 1$ = the remainder when 0^1 is divided by 5 = 0

Thus $0 \odot 1 = 0$ ---------- (1)

Similar calculations yield

$0 \odot 2 = 0$ ---------- (2)

$0 \odot 3 = 0$ ---------- (3)

$0 \odot 4 = 0$ ---------- (4)

$1 \odot 0$ = the remainder when 1^0 is divided by 5 = 1

Thus, $1 \odot 0 = 1$

Similarly, $1 \odot 1 = 1$

$1 \odot 2 = 1$

$1 \odot 3 = 1$

$1 \odot 4 = 1$

To evaluate $2 \odot 0$, we have $2^0 = 1$. This, when divided by 5, gives the remainder 1. Thus,

			$2 \odot 0 = 1$
Similarly	$2^1 = 2$	which implies	$2 \odot 1 = 2$
	$2^2 = 4$	which implies	$2 \odot 2 = 4$
	$2^3 = 8$	which implies	$2 \odot 3 = 3$
	$2^4 = 16$	which implies	$2 \odot 4 = 1$
Also	$3^0 = 1$	which implies	$3 \odot 0 = 1$
	$3^1 = 3$	which implies	$3 \odot 1 = 3$
	$3^2 = 9$	which implies	$3 \odot 2 = 4$
	$3^3 = 27$	which implies	$3 \odot 3 = 2$
	$3^4 = 81$	which implies	$3 \odot 4 = 1$
And	$4^0 = 1$	which implies	$4 \odot 0 = 1$
	$4^1 = 4$	which implies	$4 \odot 1 = 4$
	$4^2 = 16$	which implies	$4 \odot 2 = 1$
	$4^3 = 64$	which implies	$4 \odot 3 = 4$
	$4^4 = 256$	which implies	$4 \odot 4 = 1$

Thus the table for the binary operation will be

$\odot$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	1	1	1
2	1	2	4	3	1
3	1	3	4	2	1
4	1	4	1	4	1

Table 1.3

Example 7. Let $A = \{0,1,2,3,4\}$ and a binary operation $\oplus : A \times A \longrightarrow A$ be defined by $(x,y) \longmapsto x \oplus y = r$, where r is the remainder when $x + y$ is divided by 5. (Here + is the usual addition). Complete the following Cayley's table.

$\oplus$	0	1	2	3	4
0	0				
1		2			
2			4		
3				1	
4					3

Table 1. 4

$x \oplus y$ = the remainder when $x+y$ is divided by 5. We will evaluate $0 \oplus 1$ first. $0 \oplus 1$ = the remainder when $0+1$ is divided by 5 = 1.

Thus $0 \oplus 1 = 1$

Similar calculation yields $0 \oplus 2 = 2$

$0 \oplus 3 = 3$

$0 \oplus 4 = 4$

We notice that $1 + 0 = 1$ and it gives the remainder 1 when divided by 5.

Thus $1 \oplus 0 = 1$

Similarly $1 \oplus 1 = 2$

$1 \oplus 2 = 3$

$1 \oplus 3 = 4$

$1 \oplus 4 = 0$, as $1 + 4 = 5$ leaves remainder 0 when divided by 5.

Calculating this way, we get

$2 \oplus 0 = 2$

$2 \oplus 1 = 3$

$2 \oplus 2 = 4$

$2 \oplus 3 = 0$

$2 \oplus 4 = 1$, as $2 + 4 = 6$ leaves remainder 1 when divided by 5.

$3 \oplus 0 = 3$

$3 \oplus 1 = 4$

$3 \oplus 2 = 0$

$3 \oplus 3 = 1$

$3 \oplus 4 = 2$

$4 \oplus 0 = 4$

$4 \oplus 1 = 0$

$4 \oplus 2 = 1$

$4 \oplus 3 = 2$

$4 \oplus 4 = 3$

Thus the completed Cayley's table is

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 1.5

[This kind of binary operation together with set A is called 5- hour clock arithmetic or arithmetic modulo 5.]

Exercise 1.9

1. Let $A = \{ 0,1,2 \}$ and the mapping Θ be defined by

   $(x,y) \longmapsto x \Theta y = x + 3y.$

   Is the mapping a binary operation ?

2. Let J^+ be the set of all positive integers. Is the function Θ defined by

   $a \Theta b = a(2a + b),\ a, b \in J^+$, a binary operation?

   If it is a binary operation, find

   (i) $2 \Theta 3$ (ii) $3 \Theta 2$ (iii) $(2 \Theta 3) \Theta 4$

   (iv) $2 \Theta (3 \Theta 4)$ (v) Is $2 \Theta 3 = 3 \Theta 2$?

   (v) (vi) Is $(2 \Theta 3) \Theta 4 = 2 \Theta (3 \Theta 4)$?

3. Show that the mapping Θ defined by $x \Theta y = xy - x - y$ is a binary operation on the set R of real numbers.

   (a) Is the binary operation commutative?

   [i.e., Is $x \Theta y = y \Theta x$, for all $x, y \in R$?]

   (b) Find $(2 \Theta 3) \Theta 4$ and $2 \Theta (3 \Theta 4)$. Are they equal?

   (c) Is the binary operation associative?

   [i.e., Is $(x \Theta y) \Theta z = x \Theta (y \Theta z)$, for all $x,\ z \in R$?]

4. The operation Θ is defined by $x \Theta y = x^2 - 4xy - 5y^2$. Calculate $5 \Theta 4$. Find the possible values of x such that $x \Theta 2 = 28$.

5. A binary operation Θ on R is defined by $x \Theta y = x(3x + 2y)$, for all real number x and y. Find $(1 \Theta 2) \Theta 3$. Find the possible values of x such that $x \Theta 3x = 36$.

6. The binary operation Θ on R is defined by $x\Theta y = \dfrac{x^2 + y^2}{2} - xy$, for all real numbers x and y. Show that the operation is commutative, and find the possible values of a such that $a \Theta 2 = a + 2$.

7. Copy and complete the following Cayley tables. (Which of the system has closure property?)

(a)

+	E	O
E		
O		E

E denotes an even number
O denotes an odd number
+ denotes ordinary addition.

(b)

$\oplus_3$	0	1	2
0			
1			
2			

$\oplus_3$ denotes addition in 3- hour clock arithmetic (based on a clock with the numerals 0,1,2).

(c)

$\otimes_5$	0	1	2	3	4
0	0				
1	0				
2	0				
3	0				
4	0				

$\otimes_5$ is multiplication in 5-hour clock arithmetic. (Notice that the entries are the remainders on division by 5.)

(d)

∪	∅	A	B	C
∅	∅			
A				C
B		C		
C				C

A = { 1,2,3} , B = {3,4,5}
C = {1,2,3,4,5}.
∪ is the operation of union.

(e)

∩	∅	A	B	C
∅	∅			
A			{3}	
B				
C				

A,B,C are the sets in question (d)
∩ is the operation of intersection.

8. By producing one counter-example in each case, show that none of the following are associative.

(a) operation $-$ on Z (b) operation $+$ on the set $\{x \mid x \in Q \text{ and } x > 0\}$

(c) operation Θ on N, where $a \Theta b = a^b$.

SUMMARY

Functional Notation

If $f : A \longrightarrow B$ where $f : x \longmapsto x^2 + x + 1$, we write $f(x) = x^2 + x + 1$. This expression gives " the formula for the function f ".

It is a convenient way to describe the rule which tells us what the image of each element of the domain is.

The image of $k \in A$ is $f(k)$, and it is computed by substituting k for x in the "formula".

One - to - one correspondence

Let $f : A \longrightarrow B$. If each element of B is related to exactly one element on A, then f is called one - to - one correspondence between A and B.

Composite of f and g .

$(g \circ f)(x) = g(f(x))$

Identity function

$I : A \longrightarrow A$, defined by $I(x) = x$ for every $x \in A$, is the identity function on the set A.

Associative Law

$h \circ (g \circ f) = (h \circ g) \circ f$ holds true.

Commutative Law

The composition of functions does not; in general, obey the commutative law.

Inverse function

A function $f : A \longrightarrow B$ has the inverse function $g : B \longrightarrow A$ if each element of B is the image of a unique element of A.

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

Binary Operation

Binary operation on a set A is a function from A $\times$ A into A where the range is a proper subset of A. The property that the range is a proper subset of the given set is known as the closure property.

In symbol, we write " Θ " for the function and thus we have the function " Θ " from A$\times$ A into A.

We use the notation x Θ y for image of the ordered pair (x,y).

Thus Θ (x,y) = x Θ y.

Cayley tables are widely used for binary operations.

A binary operation is commutative if x Θ y = y Θ x.

A binary operation is associative if (x Θ y) Θ z = x Θ (y Θ z).

CHAPTER 2

The Remainder Theorem and the Factor Theorem

2.1 The Remainder Theorem

We know how to divide a polynomial expression $a_0x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ by polynomials of the first degree, such as $x-1$, $x+2$, $2x+3$, $x-\frac{1}{2}$ and so on. The remainder is always a constant. The remainder does not depend on the value of x. Our aim is to find the remainder without doing the actual division.

We will use the functional notation f(x) to denote a polynomial in x. When we deal with more than one polynomial, we may denote the polynomials by P(x), Q(x), g(x), etc.

Theorem

If a polynomial f(x) is divided by $x-k$, the remainder is f(k).

Proof : Let Q(x) be the quotient and let R be the remainder when f(x) is divided by $x-k$.

R does not depend on the value of x.

We have $f(x) = (x-k)\,Q(x) + R$.

Substitute k for x. Then

$$\begin{aligned} f(k) &= (k-k)\,Q(k) + R \\ &= 0.Q(k) + R \quad = \quad 0 + R \\ &= R \end{aligned}$$

That is, $R = f(k)$.

This theorem is known as **the remainder theorem.**

Example 1. Find the remainder when $x^8 + 2x - 5$ is divided by $x-1$.

Solution.

Let $f(x) = x^8 + 2x - 5$.

when f(x) is divided by $x-1$,

the remainder $= f(1) = 1^8 + 2\times 1 - 5 = -2$

Example 2. Find the remainder when $x^7 + 3x^2 - 5$ is divided by $x+1$.

Solution

Let $f(x) = x^7 + 3x^2 - 5$.

We see that, $x + 1 = x - (-1)$

When f(x) is divided by x + 1,

the remainder = $f(-1) = (-1)^7 + 3(-1)^2 - 5 = -3$

Example 3. When the polynomial $x^3 - 3x^2 + kx + 7$ is divided by $(x + 3)$, the remainder is 1. Find the value of k.

Solution

Let $f(x) = x^3 - 3x^2 + kx + 7$.

When f(x) is divided by $(x + 3)$,

the remainder = $f(-3) = (-3)^3 - 3(-3)^2 + k(-3) + 7$

$= -27 - 27 - 3k + 7 = -47 - 3k$

By using the given condition,

$$\begin{aligned} -47 - 3k &= 1 \\ 3k &= -48 \\ \therefore\ k &= -16. \end{aligned}$$

2.2 Extension of the Remainder Theorem

Let us consider the division of f(x) by $(ax - b)$. If f(x) is divided by $(x - \frac{b}{a})$, the remainder is $f(\frac{b}{a})$.

Let Q(x) be the quotient. Then

$$\begin{aligned} f(x) &= (x - \frac{b}{a})\,Q(x) + f(\frac{b}{a}) = (\frac{ax - b}{a})\,Q(x) + f(\frac{b}{a}) \\ &= (ax - b)\,\frac{Q(x)}{a} + f(\frac{b}{a}) \end{aligned}$$

$\therefore$ when f(x) is divided by $(ax - b)$, the remainder is $f(\frac{b}{a})$.

The quotient in this case is $\frac{Q(x)}{a}$.

Example 4. Find the remainder when $2x^3 + x^2 - 5x + 3$ is divided by $2x + 1$.

Solution

Let $f(x) = 2x^3 + x^2 - 5x + 3$.

We see that $2x + 1 = 2x - (-1)$

$\therefore$ remainder $= f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + (-\frac{1}{2})^2 - 5(-\frac{1}{2}) + 3$

$= -\frac{1}{4} + \frac{1}{4} + \frac{5}{2} + 3 = 5\frac{1}{2}$

Exercise 2.1

Using the remainder theorem, find the remainder when :

1. $2x^2 - 13x + 10$ is divided by $x-3$.
2. $x^3 - 3x^2 + 5x - 9$ is divided by $x-2$.
3. $x^3 + 4x^2 + 6x + 5$ is divided by $x+2$.
4. $x^6 - x^3 - 1$ is divided by $x + 2$.
5. $9x^2 + 6x - 10$ is divided by $3x + 1$.
6. $3x^3 + 5x^2 - 11x + 8$ is divided by $3x - 1$
7. $6x^3 + x^2 + 1$ is divided by $2x - 3$.
8. $3(x + 4)^2 - (1 - x)^3$ is divided by x.
9. $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $2x + 1$.
10. The polynomial $x^3 + ax^2 + bx - 3$ leaves a remainder of 27 when divided by $x - 2$ and a remainder of 3 when divided by $x + 1$. Calculate the remainder when the polynomial is divided by $x - 1$.
11. The expression $6x^2 - 2x + 3$ leaves a remainder of 3 when divided by $x - p$. Determine the values of p.
12. The expressions $x^3 - 7x + 6$ and $x^3 - x^2 - 4x + 24$ have the same remainder when divided by $x + p$. Find the possible values of p.
13. Given that the expression $x^3 - ax^2 + bx + c$ leaves the same remainder when divided by $x + 1$ or $x - 2$, find a **in terms** of b.
14. Given that the remainder when $x^3 - x^2 + ax$ is divided by $x + a$ where $a > 0$, is twice the remainder when it is divided by $x - 2a$, find the value of a.

15. The remainder when ax^3+bx^2+2x+3 is divided by $x-1$ is twice that when it is divided by $x+1$, show that $b=3a+3$.

16. The remainder when x^4+3x^2-2x+2 is divided by $x+a$ is the square of the remainder when x^2-3 is divided by $x+a$. Calculate the possible values of a.

17. The expression ax^3-x^2+bx-1 leaves the remainders of -33 and 77 when divided by $x+2$ and $x-3$ respectively. Find the values of a and b and the remainder when divided by $x-2$.

2.3 The Factor Theorem

Theorem

Let f(x) be a polynomial. Then $(x-k)$ is a factor of f(x) if and only if $f(k)=0$.

Proof: Suppose that $f(k)=0$.

By the remainder theorem, f(k) is the remainder when f(x) is divided by $(x-k)$.

So we have the remainder 0.

Thus f(x) is divisible by $(x-k)$. That is $(x-k)$ is a factor of f(x).

Conversely, suppose that $(x-k)$ is a factor of f(x).

Then $f(x) = (x-k)\,Q(x)$ for some polynomial Q(x).

$\therefore\ f(k)=(k-k)Q(k) = 0\,.Q(k) = 0.$

Example 1. Determine whether or not $x+1$ is a factor of the following polynomials.

(a) $3x^4+x^3-x^2+3x+2$ (b) $x^6+2x(x-1)-4.$

Solution (a) Let $f(x) = 3x^4+x^3-x^2+3x+2.$

$$f(-1) = 3(-1)^4+(-1)^3-(-1)^2+3(-1)+2$$
$$= 3(1)+(-1)-1-3+2 = 3-1-1-3+2 = 0$$

$\therefore\ x+1$ is a factor of f(x).

(b) Let $g(x) = x^6+2x(x-1)-4$

$$g(-1)=(-1)^6+2(-1)(-1-1)-4 = 1+2(-1)(-2)-4$$
$$=1+4-4=1$$

So, $g(-1)\neq 0$.

$\therefore\ x+1$ is not a factor of g(x).

Example 2. Find what values p must have in order that $(x - p)$ may be a factor of $4x^3 - (3p + 2)x^2 - (p^2 - 1)x + 3$.

Solution

Let $f(x) = 4x^3 - (3p + 2)x^2 - (p^2 - 1)x + 3$

$(x - p)$ is a factor of $f(x)$ only if $f(p) = 0$

That is, $4p^3 - (3p + 2)p^2 - (p^2 - 1)p + 3 = 0$

or $\qquad 2p^2 - p - 3 = 0$

$(p + 1)(2p - 3) = 0$

$\therefore \quad p = -1 \text{ or } p = \frac{3}{2}$

Example 3. Find the factors of $x^3 - 3x^2 - 4x + 12$.

Solution

Let $\quad f(x) = x^3 - 3x^2 - 4x + 12$

We shall try the integers which divides 12,

namely $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

$f(1) = 1 - 3 - 4 + 12 \neq 0$

$f(-1) = (-1)^3 - 3(-1)^2 - 4(-1) + 12 = -1 - 3 + 4 + 12 \neq 0$

$f(2) = (2)^3 - 3(2)^2 - 4(2) + 12 = 8 - 12 - 8 + 12 = 0$

By the factor theorem, $(x - 2)$ is a factor of $f(x)$.

The other factors can be found by actual division as follows :

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\
\hline
x - 2 & x^3 - 3x^2 - 4x + 12 \\
 & x^3 - 2x^2 \\
\hline
 & \quad -x^2 - 4x \\
 & \quad -x^2 + 2x \\
\hline
 & \qquad -6x + 12 \\
 & \qquad -6x + 12 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$\therefore \quad f(x) = (x - 2)(x^2 - x - 6) = (x - 2)(x - 3)(x + 2)$

$\therefore \quad$ the factors are $(x - 2)$, $(x - 3)$ and $(x + 2)$

Example 4 . If the equations $ax^3 + 4x^2 - 5x - 10 = 0$ and $ax^3 - 9x - 2 = 0$ have a common root, then show that $a = 2$ or 11.

Solution

Let c be a common root of

$P(x) = ax^3 + 4x^2 - 5x - 10 = 0$ and $Q(x) = ax^3 - 9x - 2 = 0.$

Then $(x - c)$ is a factor of both $P(x)$ and $Q(x)$.

That is , $P(c) = 0$ and $Q(c) = 0$

Thus, $ac^3 + 4c^2 - 5c - 10 = 0$

and $ac^3 - 9c - 2 = 0$

Subtracting, we get

$$4c^2 + 4c - 8 = 0$$

$$\therefore \quad c^2 + c - 2 = 0$$

$$\therefore \quad (c + 2)(c - 1) = 0$$

$$\therefore \quad c + 2 = 0 \quad \text{or} \quad c - 1 = 0$$

$$\therefore \quad c = -2 \quad \text{or} \quad c = 1$$

If $c = -2$, then $a(-2)^3 - 9(-2) - 2 = 0$

$$\therefore \quad -8a + 18 - 2 = 0$$

$$\therefore \quad a = 2$$

If $c = 1$, then $a(1)^3 - 9(1) - 2 = 0$

$$\therefore \quad a - 9 - 2 = 0$$

$$\therefore \quad a = 11$$

Exercise 2.2

Show by means of the factor theorem that :

1. $x - 4$ is a factor of $2x^4 - 9x^3 + 5x^2 - 3x - 4$.
2. $2x - 1$ is a factor of $2x^3 + x^2 + 5x - 3$.
3. Find the factors of $3x^3 - 4x^2 - 3x + 4$.
4. Find the factors of $2x^3 - 5x^2 + 4x - 1$.
5. Find the factors of $2x^3 + x^2 - 13x + 6$.
6. Find the factors of $x^4 - 14x^3 + 71x^2 - 154x + 120$.

7. Find the factors of $x^4 - 9x^2 - 4x + 12$.

8. Solve the equations :

   (a) $x^3 - 8x^2 - 31x - 22 = 0$

   (b) $6x^3 + x^2 - 19x + 6 = 0$

   (c) $x^3 - 6x^2 + 11x = 6$

   (d) $x^4 - 4x^3 - x^2 + 16x = 12$

9. Find the value of p if $4x^4 - 12x^3 + 13x^2 - 8x + p$ is divisible by $(2x - 1)$

10. Find what value p must have in order that $x+2$ may be a factor of $2x^3+3x^2+px-6$. Find the other factors.

11. If $x^2 + 2x - 3$ is a factor of $f(x) = x^4 + 2x^3 - 7x^2 + ax + b$, find a and b, hence factorize f(x) completely.

12. Find the value of p and q if $6x^3 + 13x^2 + px + q$ is exactly divisible by $2x^2+7x-4$. Show also that $(3x - 4)$ is a factor of the given polynomial.

13. Prove that $(x - 1)$ is a factor of $2x^3 - 13x^2 + 23x - 12$, and find the other factors. Hence solve $2x^3 - 13x^2 + 23x - 12 = 0$.

14. The polynomial $ax^3 + bx^2 - 5x + 2a$ is exactly divisible by $x^2 - 3x - 4$. Calculate the value of a and b, and factorize the polynomial completely.

15. $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - 5x + 3a$ have a common factor $x + 2$. Find a and b.

16. The expression $x^3 + ax^2 + bx + 3$ is exactly divisible by $x + 3$ but it leaves a remainder of 91 when divided by $x - 4$. What is the remainder when it is divided by $x + 2$?

17. The expression $px^3 - 5x^2 + qx + 10$ has factor $2x - 1$ but leaves a remainder of -20 when divided by $x + 2$. Find the values of p and q and factorize the expression completely.

18. $x + 2$ is a factor of $f(x) = a(x - 1)^2 + b(x - 1) + 9$. The remainder when f(x) is divided by $x + 1$ is -11. Find the value of a and b.

19. Show that the expression $x^3 + (k - 2)x^2 + (k - 7)x - 4$ has a factor $x + 1$ for all values of k. If the expression also has a factor $x + 2$, find the value of k and the third factor.

20. Given $f(x) = x^3 + px^2 - 2x + 4\sqrt{3}$ has a factor $x + \sqrt{2}$, find the value of p. Show that $x - 2\sqrt{3}$ is also a factor and solve the equation $f(x) = 0$.

21. Given that $2x^2 - x - 1$ is a factor of $ax^4 + x^3 - bx^2 + 5x + 6$, find the values of a and b.

22. Given that $kx^3 + 2x^2 + 2x + 3$ and $kx^3 - 2x + 9$ have a common factor, what are the possible values of k?

23. Given $f(x) = 2x^3 + ax^2 - 7a^2x - 6a^3$, determine whether or not $x - a$ and $x + a$ are factors of f(x). Hence find, in terms of a, the roots of $f(x) = 0$.

SUMMARY

1. The Remainder Theorem.

   If a polynomial f(x) is divided by $(x - k)$, the remainder is f(k).

   Extension: If a polynomial f(x) is divided by $(ax - b)$, the remainder is $f(\frac{b}{a})$.

2. The Factor Theorem.

   Let f(x) be a polynomial. Then $(x - k)$ is a factor of f(x) if and only if $f(k) = 0$.

CHAPTER 3

The Binomial Theorem

3.1 Binomial Expansion

An expression of the from $(x + y)$ raised to any power is **called** a **binomial.**

For example, $(x + y)^7$, $(x + y)^{-4}$, $(x - y)^{3/4}$ are binomials.

We will consider the case where the index is a positive integer.

By long multiplication,

$$
\begin{aligned}
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2+2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4 \\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5
\end{aligned}
$$

Notice the following important features in the above expansions.

1. When the power of the binomial is n, there are n + 1 terms.
2. The sum of the powers of x and y in every term is equal to the **power of the** binomial.
3. When the terms are arranged as shown above, the powers **of x are** in descending order while the powers of y are in ascending order.
4. When the powers are arranged in the order shown above, the **coefficients form** a pattern as shown below.

Binomial	Coefficients
$(x+y)^1$	1 1
$(x+y)^2$	1 2 1
$(x+y)^3$	1 3 3 1
$(x+y)^4$	1 4 6 4 1
$(x+y)^5$	1 5 10 10 5 1

5. Each coefficient of a line is obtained by adding the two coefficients on either side of it in the line above.

$$
\begin{array}{ccccccccccccc}
0 & & 1 & & 4 & & 6 & & 4 & & 1 & & 0 \\
 & + & & + & & + & & + & & + & & + & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 &
\end{array}
$$

The above table of coefficients for binomial expansions is called **Pascal's Triangle**, in honour of the great French mathematician, Blaise Pascal (1623 – 1662)

Example 1. Write down and simplify the expansion of $(a + 2b)^5$. Hence find the expansion of $(a - 2b)^5$.

Solution

The binomial expansion of $(a + 2b)^5$ is similar to that of $(x + y)^5$. The coefficients are obtained from Pascal's triangle while y is replaced by 2b and x by a

$$\begin{aligned}(a+2b)^5 &= a^5 + 5a^4(2b) + 10a^3(2b)^2 + 10a^2(2b)^3 + 5a(2b)^4 + (2b)^5 \\ &= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5 \\ (a-2b)^5 &= (a + (-2b))^5 \\ &= a^5+5a^4(-2b) + 10a^3(-2b)^2 + 10a^2(-2b)^3 + 5a(-2b)^4 + (-2b)^5 \\ &= a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5\end{aligned}$$

Example 2. Expand $(3a - \frac{b}{4})^4$ and simplify.

Solution

$$\begin{aligned}(3a - \frac{b}{4})^4 &= \left[3a + \left(-\frac{b}{4}\right)\right]^4 \\ &= (3a)^4 + 4(3a)^3(-\frac{b}{4}) + 6(3a)^2(-\frac{b}{4})^2 + 4(3a)(-\frac{b}{4})^3 + (-\frac{b}{4})^4 \\ &= 81a^4 - 27a^3b + \frac{27}{8}a^2b^2 - \frac{3}{16}ab^3 + \frac{b^4}{256}\end{aligned}$$

Example 3. Find, in ascending powers of x, the first three terms in the expansion of

(1) $(1 + 2x)^5$ (2) $(3 - x)^4$

Hence find the coefficient of x^2 in the expansion of $(1 + 2x)^5(3 - x)^4$.

Solution

(1) $$\begin{aligned}(1+2x)^5 &= 1^5 + 5(1)^4(2x) + 10(1)^3(2x)^2 + \ldots \\ &= 1 + 10x + 40x^2 + \ldots\end{aligned}$$

$$\begin{aligned}
(2) \quad (3-x)^4 &= [3+(-x)]^4 \\
&= 3^4 + 4(3)^3(-x) + 6(3)^2(-x)^2 + \dots \\
&= 81 - 108x + 54x^2 + \dots \\
(1+2x)^5(3-x)^4 &= (1 + 10x + 40x^2 + \dots)(81 - 108x + 54x^2 - \dots) \\
\text{coefficient of } x^2 &= 54 + 10(-108) + 40(81) = 2214
\end{aligned}$$

Exercise 3.1

1. Expand the followings

(i) $(1+2x)^4$ (ii) $(2x-y)^5$ (iii) $(a+2b)^3$

(iv) $(2+\frac{x}{2})^5$ (v) $(\frac{1}{2}x+\frac{1}{3}y)^4$ (vi) $(1-\frac{3}{x})^4$

(vii) $(x-\frac{1}{y})^3$ (viii) $(\frac{a}{2}-\frac{3}{b})^5$ (ix) $(3y-\frac{2}{x})^3$

2. Find, in ascending powers of **x**, the first three terms of the expansion of

(i) $(1+2x)^5$ (ii) $(3-x)^5$

Hence obtain the coefficient of x^2 in the expansion $(3+5x-2x^2)^5$.

3. Find, in ascending powers of x, the first 3 terms of the expansions $(1-2x)^4$ and $(2+x^2)^5$. Hence find the coefficient of x^2 in the expansion of $(1-2x)^4(2+x^2)^5$.

4. Find, in ascending powers of x, the first three terms of the expansions of

(i) $(1+2x)^4$ (ii) $(2-\frac{1}{2}x)^5$

Hence find the coefficient of x^2 in the expansion of $(1+2x)^4(2-\frac{1}{2}x)^5$.

5. Expand, in descending powers of x, the expansions of $(2x+\frac{1}{2x})^5$ and $(2x-\frac{1}{2x})^5$. Hence, or otherwise (i) simplify $(2x+\frac{1}{2x})^5+(2x-\frac{1}{2x})^5$ (ii) find the coefficient of x^2 in the expansion $(2x+\frac{1}{2x})^5 \cdot (2x-\frac{1}{2x})^5$.

3.2 The Binomial Theorem

One defect of the Pascal Triangle is that the coefficient in any line cannot be obtained unless those in the preceeding lines have been found. This defect was overcome by Pascal when he expressed the coefficients as fractions whose numerators and denominators are formed by factors obeying a simple rule.

For example, the coefficients of $(x + y)^4$ are 1, 4, 6, 4, 1 which can be expressed as

$$1, \quad \frac{4}{1}, \quad \frac{4.3}{1.2}, \quad \frac{4.3.2}{1.2.3}, \quad \frac{4.3.2.1}{1.2.3.4}$$

Similarly, the coefficients of $(x + y)^5$ are

1, 5, 10, 10, 5, 1 which can be expressed as

$$1, \quad \frac{5}{1}, \quad \frac{5.4}{1.2}, \quad \frac{5.4.3}{1.2.3}, \quad \frac{5.4.3.2.1}{1.2.3.4.5}$$

In general, if n is a positive integer,

$$(x + y)^n = x^n + \frac{n}{1}x^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + ---$$
$$+ \frac{n(n-1)(n-2)---(n-r+1)}{1.2.3---r}x^{n-r}y^r + ---$$
$$+ \frac{n(n-1)(n-2)---2}{1.2.3---(n-1)}x\,y^{n-1} + y^n$$

The coefficient written in the above expansion can be denoted by ${}^nC_r$.

Thus $\quad 1 = {}^nC_0, \quad n = {}^nC_1, \quad \frac{n(n-1)}{1.2} = {}^nC_2, \quad \frac{n(n-1)(n-2)}{1.2.3} = {}^nC_3$

and $\quad \frac{n(n-1)(n-2)---(n-r+1)}{1.2.3---r} = {}^nC_r.$

Thus the binomial expansion can be expressed as

$$(x+y)^n = {}^nC_0\,x^ny^0 + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}y^2 + --- +$$
$${}^nC_r\,x^{n-r}y^r + --- + {}^nC_{n-1}\,xy^{n-1} + {}^nC_nx^0y^n$$

This is known as the **Binomial Theorem.**

This theorem is true for all values of x and y when n is a positive integer.

Note : The term ${}^nC_rx^{n-r}y^r$ which is the $(r+1)^{th}$ term in the expansion is called the **general term.**

Special Case,

$$(1+x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \cdots + {}^nC_{n-1}\, x^{n-1} + x^n$$

Properties of the binomial coefficients

Consider the expansion of $(x+y)^n$, where n is a positive integer .

(1) The binomial coefficients are all integers .

(2) The coefficients of terms equidistant from the beginning and end of the expansion

are equal . i.e. ${}^nC_0 = {}^nC_n = 1, {}^nC_1 = {}^nC_{n-1} = n, \ldots, {}^nC_r = {}^nC_{n-r}$

We will prove that ${}^nC_r = {}^nC_{n-r}$

Proof : $$\frac{{}^nC_r}{{}^nC_{n-r}} = \frac{\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdots r}}{\dfrac{n(n-1)(n-2)\cdots(n-(n-r)+1)}{1\cdot 2\cdot 3\cdots (n-r)}}$$

$$= \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r} \times \frac{1\cdot 2\cdot 3\cdots(n-r)}{n(n-1)\cdots(r+1)}$$

$$= \frac{1\cdot 2\cdot 3\cdots(n-r)(n-r+1)\cdots(n-1)n}{1\cdot 2\cdot 3\cdots r(r+1)\cdots(n-1)n} = \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n}$$

$$= 1$$

$$\therefore {}^nC_r = {}^nC_{n-r}$$

$$\therefore \quad {}^{10}C_8 = {}^{10}C_2 = \frac{10\times 9}{1\times 2} = 45$$

$${}^{15}C_{12} = {}^{15}C_3 = \frac{15\times 14\times 13}{1\times 2\times 3} = 455$$

Example 1. Find in ascending powers of x, the first three terms in the expansion of

(i) $(1+2x)^5$ (ii) $(2-\frac{1}{2}x)^6$

Hence find the coefficient of x^2 in the expansion of

$(1+2x)^5 (2-\frac{1}{2}x)^6$

Solution (i) $$(1+2x)^5 = 1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + \cdots$$
$$= 1 + 10x + 40x^2 + \cdots$$

(ii) $(2-\frac{1}{2}x)^6 = {}^6C_0(2)^6 + {}^6C_1(2)^5(-\frac{1}{2}x) + {}^6C_2(2)^4(-\frac{1}{2}x)^2 + \cdots$

$= 64 - 96x + 60x^2 + \cdots$

$(1+2x)^5\ (2-\frac{1}{2}x)^6 = (1 + 10x + 40x^2 + \cdots)(64 - 96x + 60x^2 + \cdots)$

coefficient of $x^2 = 60 + 10(-96) + 40(64) = 1660$

Example 2. In the binomial expansion of $(3 + kx)^9$, the coefficient of x^3 and of x^4 are equal. Calculate the value of k.

Solution $(r+1)^{th}$ term $= {}^9C_r\,(3)^{9-r}(kx)^r = {}^9C_r\,3^{9-r}k^rx^r$

To get coefficient of x^3 and x^4, put $r = 3$ and 4 respectively.

coefficient of $x^3 = {}^9C_3\,3^6k^3$

coefficient of $x^4 = {}^9C_4\,3^5k^4$

$\therefore\ {}^9C_3\,3^6k^3 = {}^9C_4\,3^5k^4$

$\therefore \qquad = \dfrac{{}^9C_3\,3^6}{{}^9C_4\,3^5} = 2$

Example 3. Find the term in x^4 and the term independent of x in the expansion of

$(x + \frac{1}{x})^{20}$

Solution $(r+1)^{th}$ term $= {}^{20}C_r\,x^{20-r}(\frac{1}{x})^r = {}^{20}C_r\,x^{20-2r}$

To find the term in x^4, put $20-2r = 4$

$r = 8$

$\therefore$ the term in x^4 is the $(8+1)^{th}$ term.

i.e. 9^{th} term.

The 9^{th} term $= {}^{20}C_8\,x^4 = 125970\,x^4$

To get the term independent of x, put $20 - 2r = 0$

$r = 10$

$\therefore$ the term independent of x is the $(10+1)^{th}$ term

i.e. 11^{th} term.

The 11^{th} term $= {}^{20}C_{10} = 184756$

Exercise 3.2

1. Find and simplify the coefficient of x^7 in the expansion of $(x^2+\frac{2}{x})^8$.

2. Find the term independent of x in the expansion of $(x^2+\frac{2}{x})^6$.

3. If the coefficient of x^4 in the expansion of $(3+2x)^6$ is equal to the coefficient of x^4 in expansion of $(k+3x)^6$, find k.

4. Find the coefficient of x^{-10} in the expansion $(2-\frac{1}{x^2})^8$.

5. Find the coefficients of x^2 and x^3 in the expansion of $(1-3x)^6(1+2x)^7$.

6. Find the term independent of x in the expansion of $(\frac{1}{2x^2}-x)^9$.

7. If the coefficients of x^4 and x^5 in the expansion of $(3+kx)^{10}$ are equal, find the value of k.

8. Find the coefficient of x^6 in the expansion of (a) $(2+x)^8$ (b) $(x-\frac{3}{x})^{14}$.

9. In the expansion of $(x^2+\frac{a}{x})^8$, $a \neq 0$, the coefficient of x^7 is four times the coefficient of x^{10}. Find the value of a.

10. Given that the coefficient of x^3 in the expansion of $(a+x)^5+(1-2x)^6$ is -120, calculate the possible values of a.

11. The coefficient of x^2 in the expansion to $(2x+k)^6$ is equal to the coefficient of x^5 in the expansion of $(2+kx)^8$. Find k.

12. If the 2nd and the 3rd term in $(a+b)^n$ are in the same ratio as the 3rd and 4th in $(a+b)^{n+3}$, then find n.

13. Evaluate the coefficients of x^5 and x^4 in the binomial expansion of $(\frac{x}{3}-3)^7$. Hence evaluate the coefficient of x^5 in the expansion of $(\frac{x}{3}-3)^7(x+6)$.

14. In the binomial expansion of $(1+\frac{1}{4})^n$, the 3^{rd} term is twice the 4^{th} term. Calculate the value of n.

15. In the expansion of $(2 + 3x)^n$, the coefficient of x^3 and x^4 are in the ratio 8 : 15 Find the value of n.

16. Given that the coefficient of x^2 in the expansion of $(4 + kx)(2 - x)^6$ is zero, find the values of k.

17. Given that the coefficient of x^2 in the expansion of $(1 - ax)^6$ is 60 and that $a > 0$, find the value of a.

18. Given that the coefficients of x^2 and x^3 in the expansion of $(3 + x)^{20}$ are a and b respectively, evaluate $\frac{a}{b}$.

19. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is double the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

20. Write down the third and fourth terms in the expansion of $(a + bx)^n$. If these terms are equal, show that $3a = (n - 2)bx$.

21. The first three terms in the binomial expansion of $(a + b)^n$, in ascending powers of b, are denoted by p, q and r respectively. Show that $\frac{q^2}{pr} = \frac{2n}{n-1}$.

SUMMARY

1. The Binomial Theorem:

$$(x+y)^n = x^n + \frac{n}{1}x^{n-1}y + \frac{n(n-1)}{1.2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}x^{n-r}y^r + \dots +$$
$$\frac{n(n-1)(n-2)\dots 2}{1.2.3\dots(n-1)}xy^{n-1} + y^n$$

2. The $(r + 1)^{th}$ term is ${}^nC_r x^{n-r} y^r$, where ${}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$

3. ${}^nC_r = {}^nC_{n-r}$

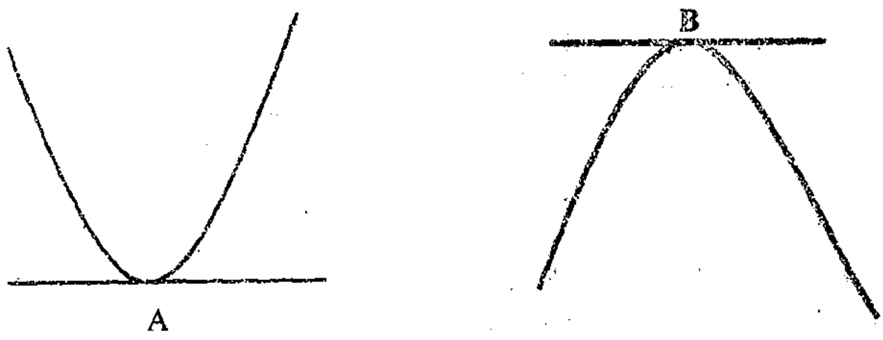
CHAPTER 4

Inequations

4.1 Quadratic Functions

The expression $f(x) = ax^2 + bx + c$ where $a \neq 0$ is called a **quadratic function.**

When the graph of the function $y = ax^2 + bx + c$ is drawn, two types of graphs are obtained depending on the value of a . These graphs are called parabolas. (see Fig.4.1)



(a) $a > 0$ Fig.4.1 (b) $a < 0$

A and B are called the vertex of the parabola. The graph is symmetrical about the line parallel to the Y– axis and passing through the vertex.

4.2 Quadratic Inequations

The open sentences $ax^2 + bx + c > 0$ and $ax^2 + bx + c < 0$ where $a \neq 0$ are **quadratic inequations in x.**

The solution set of the quadratic inequations in x can be found by

(1) Algebraic method

(2) Graphical method

Example. Find the solution set of the inequation $(x - 2)(x + 3) \geq 0$.

Solution

Method 1

We treat the function as the product of two terms $x - 2$ and $x + 3$. For the function to be positive or zero, there are two possibilities :

(a) $x - 2 \geq 0$ and $x + 3 \geq 0$

or

(b). $x - 2 \leq 0$ and $x + 3 \leq 0$

(a) $x - 2 \geq 0$ and $x + 3 \geq 0$

$x \geq 2$ $x \geq -3$

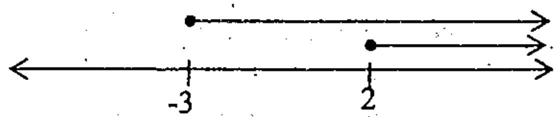


Fig. 4. 2

Referring to Fig.4.2 ,we see that all points to the right of 2 , including the point 2, will satisfy both conditions, i.e. $x \geq 2$.

(b) $x - 2 \leq 0$ and $x + 3 \leq 0$

$x \leq 2$ and $x \leq -3$

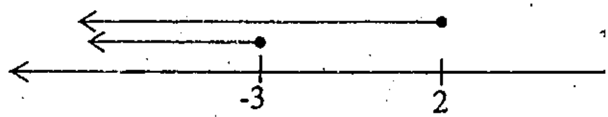


Fig.4.3

Referring to Fig.4.3 , we see that all points to the left of -3 including the point -3 , satisfy both conditions, i.e. $x \leq -3$.

$\therefore$ the solution set is $\{ x \mid x \leq -3 \text{ or } x \geq 2 \}$.

Instead of drawing a number line to find the solution set, we can either draw a table as follows.

First we find the points where the curve cuts the X-axis and determine the signs of y on the intervals by the x-intercepts points.

$y = (x - 2)(x + 3)$

When $y = 0$, $(x - 2)(x + 3) = 0$

$\therefore x = 2$ or -3

Determine the signs of y on the intervals by the points $x = 2$ and $x = -3$, as shown in the Table 4.1.

	$x < -3$	$x = -3$	$-3 < x < 2$	$x = 2$	$x > 2$
$x - 2$	$-$	$-$	$-$	0	+
$x + 3$	$-$	0	+	+	+
$y = (x - 2)(x + 3)$	+	0	$-$	0	+

Table 4.1

From the last row of table, we see that all points to the right of 2, including the point 2, will satisfy condition $y \geq 0$, (i.e. $x \geq 2$) and all points to the left of -3 including the point -3, satisfy condition $y \geq 0$, (i.e. $x \leq -3$).

$\therefore$ the solution set is $\{ x \mid x \leq -3 \text{ or } x \geq 2 \}$.

Method 2

First sketch the curve $y = (x-2)(x+3)$.

To sketch the quadratic curve, we only need to know

(i) the shape of the curve

(ii) the point(s) where it cuts the Y-axis, i.e. when $x = 0$

(iii) the points where the curve cuts the X-axis. This is given by the roots of the equation $y = 0$.

Let $y = (x-2)(x+3)$.

When $x = 0$, $y = -6$.

$\therefore$ the curve cuts the Y-axis at **(0, – 6)**.

When $y = 0$, $(x-2)(x+3) = 0$

$x - 2 = 0$ or $x + 3 = 0$

$\therefore x = 2$ or -3

$\therefore$ the curve cuts the X – axis at (2,0) and (–3,0).

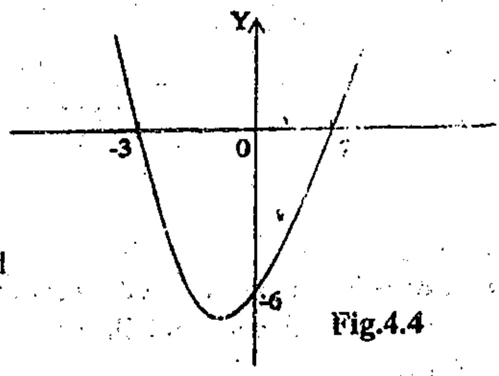


Fig.4.4

The graph of $y = (x-2)(x+2)$ is as shown in Fig.4.4.

We want $y \geq 0$, i.e. the values of x for which the curve is above or on the X – axis.

$\therefore$ the solution set is $\{ x \mid x \leq -3 \text{ or } x \geq 2 \}$.

Example 2. Use a graphical method, to find the solution set of the inequation $12 - 5x - 2x^2 \geq 0$ and illustrate it on the number line.

Solution

Let $y = 12 - 5x - 2x^2$.

When $x = 0$, $y = 12$

$\therefore$ the graph cuts the Y-axis at (0,12).

When $y = 0$, $12 - 5x - 2x^2 = 0$

$$(4 + x)(3 - 2x) = 0$$

$$x = -4 \text{ or } \frac{3}{2}$$

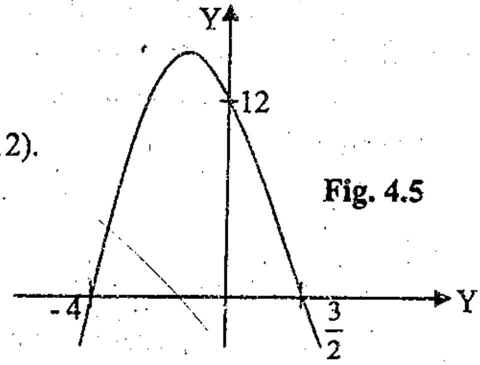


Fig. 4.5

$\therefore$ the graph cuts the X – axis at $(-4, 0)$, $(\frac{3}{2}, 0)$.

The graph of $y = 12 - 5x - 2x^2$ is as shown in Fig.4.5.

The solution set is $\{ x \mid -4 \le x \le \frac{3}{2} \}$, and its graph is as shown in Fig.4.6

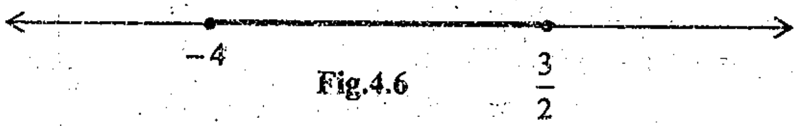


Fig.4.6

Example 3. Find the solution set of the inequation $3x^2 < x^2 - x + 3$ by algebraic method and illustrate it on the number line.

Solution

$$3x^2 < x^2 - x + 3$$

$$2x^2 + x - 3 < 0$$

$$(2x + 3)(x - 1) < 0$$

For $(2x + 3)(x - 1)$ to be negative there are two possibilities

(a) $2x + 3 < 0$ and $x - 1 > 0$

or

(b) $2x + 3 > 0$ and $x - 1 < 0$

(a) $\quad 2x + 3 < 0$ and $x > 1$

$x < -\frac{3}{2}$ and $x > 1$

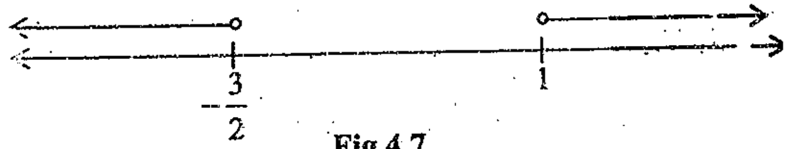


Fig.4.7

Referring to Fig.4.7 , no points satisfy both conditions.

(b) $\quad 2x + 3 > 0 \quad$ and $\quad x - 1 < 0$

$x > -\frac{3}{2}$ and $\quad x < 1$

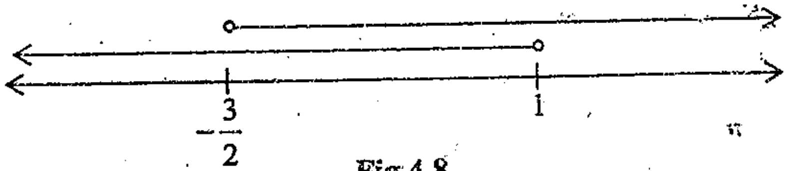


Fig.4.8

Referring to Fig.4.8 , we see that all points between $-\frac{3}{2}$ and 1 excluding $-\frac{3}{2}$ and 1 satisfy both conditions , i.e. $-\frac{3}{2} < x < 1$.

$\therefore$ the solution set is $\{ x \mid -\frac{3}{2} < x < 1 \}$ and its graph is as shown in Fig.4.9.

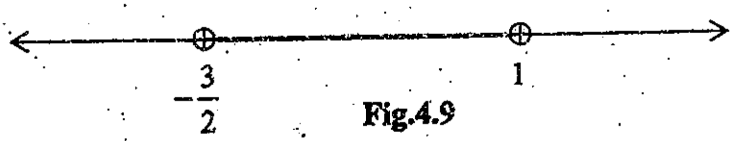


Fig.4.9

Exercise 4.1

1. (a) If $y = (1 + x)(3 - x)$, write down the roots of $(1 + x)(3 - x) = 0$ and also the value of y when $x = 0$.

   (b) Hence sketch the graph of $y = (1 + x)(3 - x)$ and use it to find the solution set of (i) $\quad (1+x)(3-x) \geq 0$

   (ii) $\quad (1+x)(3-x) < 0$.

If $y = x^2 - 4x$, find x when $y = 0$, and also find y when $x = 2$. Use a sketch graph to obtain the solution set of (a) $x^2 - 4x > 0$ (b) $x^2 - 4x \leq 0$.

(a) Solve the equation $x^2 - x - 6 = 0$.

(b) Use a graphical method to find the solution set of (i) $x^2 - x - 6 \leq 0$

(ii) $x^2 - x - 6 > 0$.

Find the solution set of each of the following inequations and illustrate it on the number line

(a) $(x - 4)(x + 7) < 0$

(b) $(3 - 4x)(4 - 3x) > 0$

(c) $12x^2 \geq 10 - 7x$

(d) $x^2 + 2 < 3x$

(e) $(x - 2)(5x - 4) + 1 > 0$

(f) $(3x - 5)^2 - 2\frac{1}{4} \geq 0$

(g) $2 + 3x > 5x^2$

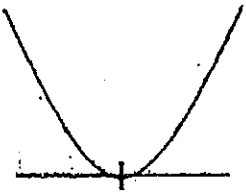
(h) $(2x + 1)(3x - 1) < 14$

(i) $x^2 + 9 > 0$

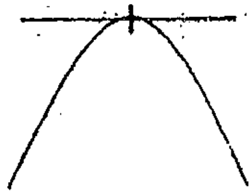
(j) $x^2 + 4 < 0$.

SUMMARY

(1) The graph of the function $y = ax^2 + bx + c$, $a \neq 0$ is as follows:



$a > 0$



$a < 0$

(2) To find the solution set of the quadratic inequations $ax^2 + bx + c < 0$ and $ax^2 + bx + c > 0$,

(i) let $y = ax^2 + bx + c$

(ii) find the points of intersection of the graph with the X-axis and Y-axis

(iii) draw the graph, and write down the solution sets.

CHAPTER 5

Sequences and Series

The word sequence is commonly used in ordinary language. For example , we may talk about a sequence of subjects that is scheduled for the final examination . What characterizes the sequence is the notion of one event following another in a definite order. There is a first subject, a second subject, a third subject, and so on. We might even give them labels.

E_1	=	Myanmar
E_2	=	English
E_3	=	Mathematics
E_4	=	Physics
E_5	=	Chemistry
E_6	=	Biology

We use a similar notation for number sequences.

5.1 Sequences

The list of numbers shown below follows a pattern

1, 4, 9, 16, 25, 36, 49

Each number in the list is called a **term.**

Thus, the first term = 1
the second term = 4
the third term = 9
the fourth term = 16
the fifth term = 25
the sixth term = 36
the seventh term = 49

The seventh term is the **last term** of the list. We can find a rule for some patterns.

'o find such a rule, we use to pair the terms with the corresponding natural numbers s follows:

$$\begin{array}{ccccccccl} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \longleftarrow & \text{natural numbers} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 & \longleftarrow & \text{terms} \end{array}$$

'rom the above correspondence, we see that each, term is the square of the orresponding natural number with which it is paired.

'his pairing can be written as a set of ordered pairs as follows.

l,1), (2,4), (3,9) ,(4,16) , (5,25) , (6,36) , (7,49) or $\{(n,n^2)\}$

/here $n \in A = \{1,2,3,4,5,6,7\}$. For each $n \in A$, n^2 is unique. Thus, if we take A as omain and the set R of real numbers as co-domain, we can define a function f from . to R by $f(n) = n^2$, $n \in A$.

ince $f(A) = \{1,4,9,16,25,36,49\}$, the range of f is the set whose elements are the :rms of the given list of numbers.

This function is called a **sequence**. The domain A of the function is a finite ɜt. Such a function is called a **finite sequence.**

'he following are some examples of finite sequences

(a) 4, 8, 12, 16, 20.

(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$.

(c) $-2, 1, \frac{1}{4}, \frac{7}{8}, -3, 9, \frac{6}{7}, 5$.

' the domain of the function is an infinite set, then such function is called an **infinite ɜquence.**

he following are some examples of infinite sequences.

(d) $3, 3^2, 3^3, \ldots, 3^n, \ldots$

(e) $2, 4, 6, 8, \ldots, 2n, \ldots$

(f) $-1, 1, -1, \ldots, (-1)^n, \ldots$

'f is the corresponding function for sequence (a) ,then $f: A \to R$, where $=\{1, 2, 3, 4, 5\}$ and is defined by $f(n) = 4n$, $n \in A$.

If g is the corresponding function for sequence (b),then $g : B \to R$, where $B = \{1,2,3,4,5,6\}$ and is defined by $g(n) = \frac{n}{n+1}$, $n \in B$.

If h is the corresponding function for sequence (c), then $h : C \to R$, where $C = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and is defined by $h(1) = -2$, $h(2) = 1$, $h(3) = \frac{1}{4}$, $h(4) = \frac{7}{8}$, $h(5) = -3$, $h(6) = 9$, $h(7) = \frac{6}{7}$, $h(8) = 5$.

Notice that there is no general rule for sequence (c) as in sequences (a) and (b).

If f represents a function for sequence (d), then $f : N \to R$ defined by $f(n) = 3^n$, $n \in N$, where N is the set of natural numbers.

If g represents a function for sequence (e), then $g : N \to R$ given by $g(n) = 2n$, $n \in N$.

If h represents a function for sequence (f), then $h: N \to R$ defined by $h(n) = (-1)^n$, $n \in N$.

Notice that the domain of the function corresponding to a finite sequence is the set of a part of the natural numbers, whereas the domain of the function corresponding to an infinite sequence is the set of all natural numbers. Thus we can define a sequence as follows.

Definition

A **sequence** is a function whose domain is either the set of all or part of the natural numbers. The values of the function are called the **terms** of the sequence. The value of the function corresponding to the number n of the domain is called the **n th term** or the **general term** of the sequence.

For these special functions called sequences, it is customary to write u_n instead of $u(n)$ for the value of the function corresponding to the natural number n. We list the values in order as $u_1, u_2, u_3, ---, u_n$ ----

Example 1. Find the first four terms of the sequence whose general term $u_n = n + 3$.

Solution

$$u_n = n + 3$$
$$u_1 = 1 + 3 = 4$$
$$u_2 = 2 + 3 = 5$$
$$u_3 = 3 + 3 = 6$$
$$u_4 = 4 + 3 = 7$$

[ence, the first four terms are 4,5,6,7.

From the above example, it is clear that a sequence can be completely etermined if the general term of a sequence is known. Conversely, if the terms of a equence follow a certain pattern and if a sufficient number of terms is known, the eneral term of the sequence can be determined. Consider the following sequences.

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

(ii) $2, 4, 6, 8, \dots$

(iii) $4, 12, 36, 108, \dots$

he sequence (i) may be rewritten as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

$\therefore$ the general term is $\frac{1}{n}$.

he sequence (ii) may be rewritten as $2\times1, 2\times2, 2\times3, 2\times4, \dots$

$\therefore$ the general term is 2n.

he sequence (iii) may be rewritten as $4\times3^0, 4\times3^1, 4\times3^2, 4\times3^3, \dots$

$\therefore$ the general term is $4\times3^{n-1}$.

A sequence may also be described by the rule of formation being defined ecursively using one or more of the earlier term of the sequence.

Example 2. Find the sequence whose first term is 1 and $u_n = 2u_{n-1}$.

Solution

$$u_1 = 1$$
$$u_n = 2u_{n-1}$$
$$u_2 = 2u_1 = 2$$
$$u_3 = 2u_2 = 4$$
$$u_4 = 2u_3 = 8 \quad \text{and so on.}$$

$\therefore$ the required sequence is 1, 2, 4, 8, ...

5.2 Series

A series is the indicated sum of the terms in a sequence. That is, associated with any sequence $u_1, u_2, u_3, \dots, u_n, \dots$

$u_1 + u_2 + u_3 + \dots + u_n + \dots$ is called a **series**.

For example, given a finite sequence 4, 7, 10, ..., 3n + 1, we may form a series

$4 + 7 + 10 + \ldots + (3n + 1)$.

Exercise 5.1

1. Find the first four terms of the sequences given by the rule of formation.

(a) $u_n = 3n$ (b) $u_n = n^2$

(c) $u_n = n + 1$ (d) $u_n = n(n + 1)$

(e) $u_n = 2^n$ (f) $u_n = \dfrac{n}{n+2}$

2. Write down the next two terms of each of the following sequences and determine the n^{th} term of each sequence.

(i) 2, 4, 8, 16, --- (ii) 1, 4, 9, 16, ---

(iii) 3, 6, 9, 12, --- (iv) 3, 8, 13, 18, ---

(v) $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, 3\sqrt{6}$, --- (vi) $-2, 2, -2, 2$, ---

3. In each case below an initial term and a recursion formula are given. Find u_4.

(i) $u_1 = 2$, $u_n = u_{n-1} + 5$

(ii) $u_1 = 3$, $u_n = 4u_{n-1}$

(iii) $u_1 = 1$, $u_n = u_{n-1} + 9(n + 1)$

(iv) $u_1 = 3$, $u_n = 2u_{n-1} + 3$

4. (a) Write down the first four terms of the sequence defined by $u_n = 4n-3$.

(b) Which term of the sequence is 189 ?

5. (a) Write down the first four terms of the sequence defined by $u_n = n^2 + 1$.

(b) Which term of the sequence is 122 ?

6. A culture of bacteria doubles in number every hour. If there were originally ten bacteria in the culture, how many will there be after two hours ? Four hours ? n hours ?

5.3 Arithmetic Progression (A.P.)

An **arithmetic progression** is a sequence in which the difference between two consecutive terms like the nth and (n+1) th term is a constant. This constant is called the **common difference** of the progression.

The following sequences are some examples of arithmetic progressions.

(i) $3, 6, 9, 12, 15, \text{---}$

(ii) $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \text{---}$

(iii) $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, \text{---}$

The common difference in (i) is 3, in (ii) is $-\frac{1}{6}$, and in (iii) is $\frac{1}{2}$

The common difference is denoted by d.

Since $u_{n+1} - u_n = d$, $n \in$ N, if we denoted u_1 by a, then

$u_2 = a + d$, $u_3 = a + 2d$, $u_4 = a + 3d$ and so on.

Thus, the general or standard form of an arithmetic progression is

$a, a + d, a + 2d, a + 3d, \ldots$

In general, the **n th term** of an A. P. is given by

$$\mathbf{u_n = a + (n - 1) d}$$

Example 1. Find the 25 th term of the following arithmetic progressions:

(i) $3, 6, 9, 12, \ldots$ (ii) $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$

Solution

(i) $3, 6, 9, 12, \ldots$

$a = 3$, $d = 3$

$u_{25} = a + 24d = 3 + 72 = 75$

$\therefore$ the 25 th term is 75.

(ii) $1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \ldots$

$a = 1$, $d = -\frac{1}{6}$

$u_{25} = a + 24d = 1 - 4 = -3$

$\therefore$ the 25 th term is -3.

Example 2. The fifth term of an arithmetic progression is 10 while the 15 th term is 40. Write down the first 5 terms of the A.P.

Solution $u_5 = 10$, $u_{15} = 40$.

Since $u_5 = a + 4d$, $u_{15} = a + 14d$,

$$a + 4d = 10$$
$$a + 14d = 40$$

Solving these two equations, we get $d = 3, a = -2$.

$\therefore$ the first five terms of the A.P. are $-2, 1, 4, 7, 10$.

Example 3. Which term of the A.P. 6, 13, 20, 27, ... is 111 ?

Solution Let 111 be the n th term of the A.P.

$a = 6$, $d = 7$ $u_n = 111$.

But $u_n = a + (n-1)d$

$$\therefore \quad a + (n-1)d = 111$$
$$6 + (n-1)7 = 111$$
$$7n = 112$$
$$n = 16$$

$\therefore$ 111 is the 16 th term of the A.P.

Exercise 5.2

1. In each of the following A.P. find
   (a) the common difference
   (b) the 10 th term
   (c) the n th term.

   (i) $1, 3, 5, 7, ---$ (ii) $10, 9, 8, 7, ---$

   (iii) $1, 2\frac{1}{2}, 4, 5\frac{1}{2}, ---$ (iv) $20, 18, 16, 14, ---$

   (v) $-25, -20, -15, -10, ---$ (vi) $-\frac{1}{8}, -\frac{1}{4}, -\frac{3}{8}, -\frac{1}{2}, ---$

2. The fifth and tenth terms of an A.P. are 8 and -7 respectively. Find the 100th and 500th terms of the A.P.
3. The sixth term of an A.P. is 32 while the 10 th term is 48. Find the common difference and the 21 st term.
4. If 5, a, b, 71 are consecutive terms of an A.P., find the value of a and of b.

5. The four angles of a quadrilateral are in A.P. Given that the value of the largest angle is three times the value of the smallest angle, find the values of all four angles.

6. If $u_1 = 6$ and $u_{30} = -52$ in an A.P, find the common difference .

7. In an A.P. , $u_1 = 3$ and $u_7 = 39$. Find

   (a) the first five terms of the A.P.

   (b) the 20 th term of the A.P.

8. If the n th term of an A.P. 2, $3\frac{7}{8}$, $5\frac{3}{4}$, --- is equal to the n th term of an A.P. 187, $184\frac{1}{4}$, $181\frac{1}{2}$, ---, find n.

Arithmetic Mean (A.M.)

In a finite arithmetic progression , the terms between the first term and the last term are called the **arithmetic means.**

For example, in the arithmetic progression 1, 5, 9, 13, the arithmetic means are 5,9. By definition , the arithmetic mean (A.M.) of two numbers a and b is given by

$$\text{A.M.} = \frac{a+b}{2}$$

Example 4. Insert three A.M. between 7 and –5.

Solution

Let x_1, x_2, x_3 be three A.M. between 7 and –5

$\therefore$ by definition, 7, x_1, x_2, x_3, –5 is an A.P., with

$a = 7, \qquad u_5 = -5$

But $\quad u_5 = a + 4d$

$$\begin{aligned}\therefore \quad a + 4d &= -5\\ 7 + 4d &= -5\\ d &= -3\end{aligned}$$

$\therefore \quad x_1 = 4$, $x_2 = 1$, $x_3 = -2$

$\therefore$ the three A.M. are 4, 1, –2.

Sum of the first n terms of an arithmetic progression

Let a, a + d, a + 2d, a + 3d,…, a + (n – 1)d, … be the given A.P.

Let S_n denote the sum of the first n terms of the A.P.

$$S_n = a + (a + d) + (a + 2d) + \text{---} + \{a + (n - 2)d\} + \{a + (n - 1)d\}$$

If ℓ denotes the last term (i.e.n th term), then

$$S_n = a + (a + d) + (a + 2d) + \text{---} + (\ell - 2d) + (\ell - d) + \ell \qquad \text{--------- (1)}$$

Writing the A.P. in the reverse order , we have

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \text{---} + (a + 2d) + (a + d) + a \qquad \text{--------- (2)}$$

Adding equations (1) and (2) , we have

$$2S_n = \underbrace{(a + \ell) + (a + \ell) + (a + \ell) + \text{---} + (a + \ell) + (a + \ell) + (a + \ell)}_{\text{n times}}$$

$$= n(a + \ell)$$

$$\therefore S_n = \frac{n}{2}(a + \ell) \qquad (3)$$

Substitute $\ell = a + (n - 1)d$ in equation (3). we have

$$S_n = \frac{n}{2}\{a + a + (n - 1)d\}$$

$$\therefore S_n = \frac{n}{2}\{2a + (n - 1)d\} \qquad (4)$$

Both formulae (3) and (4) are useful. We normally use (3) when the number of terms n, the first term a and the last term ℓ of an A.P. are given. Formula (4) is used when the number of terms n, the first term a and the common difference d of an A.P. are given. Since S_n and S_{n-1} denotes the sum to first n terms and the sum to first n – 1 terms respectively, we have

$$u_n = S_n - S_{n-1}$$

Example 5. In an arithmetic progression 44, 40, 36, ---

(a). find the sum to first 12 terms

(b) find the sum from 13 th term to 25 th term.

Solution

$a = 44, \qquad d = -4$

(a) $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$S_{12} = \frac{12}{2}\{2(44) + (12-1)(-4)\} = 264$

(b) Let S be the required sum

Then, $S = S_{25} - S_{12}$

$S_{25} = \frac{25}{2}\{2(44) + (25-1)(-4)\} = -100$

$\therefore \quad S = -100 - 264 = -364$.

Example 6. The sum of the first 8 terms of an A.P. is 56 and the sum of the first 20 terms is 260 .. Find the first term and the common difference of the A.P.

Solution

$S_n = \frac{n}{2}\{2a + (n-1)d\}$

$S_8 = \frac{8}{2}\{2a + (8-1)d\}$

$56 = 4(2a + 7d)$

$2a + 7d = 14 \qquad (1)$

$S_{20} = \frac{20}{2}\{2a + (20-1)d\}$

$260 = 10(2a + 19d)$

$2a + 19d = 26 \qquad (2)$

Solving equations (1) and (2) , we get

$d = 1, \qquad a = 3\frac{1}{2}$.

Example 7 How many terms of the arithmetic progression 9, 7, 5, --- add up to 24 ?

Solution

Let n be the number of terms.

$a = 9, \qquad d = -2, \qquad S_n = 24$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$24 = \frac{n}{2}\{18 + (n-1)(-2)\}$$

$$n^2 - 10n + 24 = 0$$
$$(n-6)(n-4) = 0$$
$$n = 6 \quad \text{(or) } 4$$

Example 8. The sum of the first n terms of the A.P.13, $16\frac{1}{2}$, 20, --- is the same as the sum of the first n terms of the A.P.3,7,11, ---. Calculate n.

Solution

For the A.P. 13, $16\frac{1}{2}$, 20, ---

$a = 13, \quad d = 3\frac{1}{2}$.

$$S_n = \frac{n}{2}\{2a + (n-1)d\} = \frac{n}{2}\{26 + (n-1)\frac{7}{2}\}$$

For the A.P. 3, 7, 11, ---

$a = 3, \qquad d = 4$

$$S_n = \frac{n}{2}\{6 + (n-1)4\}$$

Now
$$\frac{n}{2}[26 + (n-1)\frac{7}{2}\} = \frac{n}{2}\{6 + (n-1)4\}$$

$$26 - 6 = (n-1)(4 - \frac{7}{2})$$

$$20 = (n-1)\frac{1}{2}$$

$$n = 41$$

Example 9. The sum to n terms of an A.P. is 21. The common difference is 4 and the sum to 2n terms is 78. Find the first term.

Solution

$d = 4, \qquad S_n = 21. \qquad S_{2n} = 78$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$21 = \frac{n}{2}\{2a + (n-1)4\} \qquad (1)$$

$$S_{2n} = \frac{2n}{2}\{2a + (2n-1)d\}$$

$$78 = n\{2a + (2n-1)4\} \qquad (2)$$

From equation (1)

$$42 = n\{2a + (n-1)4\}$$

From equation (2)

$$78 = n\{2a + (n-1)4 + 4n\}$$
$$= n\{2a + (n-1)4\} + 4n^2 = 42 + 4n^2$$

$$4n^2 = 36$$
$$n^2 = 9$$
$$n = 3$$

Substituting n = 3 in equation (1), we get

$$21 = \frac{3}{2}\{2a + 8\} = 3(a + 4)$$
$$a + 4 = 7$$
$$a = 3$$

Example 10 A semicircle is divided into n sectors such that the angles of the sectors form an arithmetic progression. If the smaller angle is $5°$ and the largest angle is $25°$, calculate n.

Solution

$a = 5°, \ \ell = 25°, \ S_n = 180°$

$$S_n = \frac{n}{2}\{a + \ell\}$$

$$180 = \frac{n}{2}\{5 + 25\}$$

$$15n = 180$$
$$n = 12.$$

Exercise 5.3

1. Find the sum of each of the following arithmetic progressions.

   (a) $-4, -5, -6,$ --- to 18 terms

   (b) $3, 8, 13,$ --- to 98 terms

   (c) $a-b, a-2b, a-3b,$ --- to 20 terms

   (d) $1, 3, 5,$ --- to 21 terms.

2. The third and sixth terms of an A.P. are 13 and 22 respectively, find the sum of the first n terms in terms of n.

3. Find the sum of all multiples of 7 between 400 and 500.

4. Find the sum of all odd numbers between 70 and 150.

5. Find the sum of all two – digit natural numbers which are divisible by 3.

6. How many terms of an A.P. 5, 7, 9, --- give a sum of 192 ?

7. How many terms of an A.P. 24, 20, 16, --- give a sum of 0 ?

8. The sixth term of an A.P. is 22 and the 10th term is 34. Find the sum to first 16 terms of the A.P.

9. The sum of four consecutive numbers in an A.P. is 28. The product of the second and third numbers exceeds that of the first and last by 18. Find the numbers.

10. The fourth term of an A.P. is 1 and the sum of the first 8 terms is 24. Find the sum of the first three terms of the progression .

11. In an A.P. whose first terms is -27, the tenth term is equal to the sum of the first 9 terms. Calculate the common difference.

12. If m is a positive integer, show that the sum of the A.P. $2m+1, 2m+3, 2m+5, \ldots, 4m-1$ is divisible by 3.

13. For a certain A.P. $S_n = \frac{n}{2}(3n-17)$. Calculate S_1, S_2, S_3, S_4. Hence find the first 4 terms of the corresponding sequence and a formula for the n th term.

14. How many bricks are there in a pile one brick in thickness if there are 27 bricks in the bottom row, 25 in the second row etc and 1 in the top row.

15. If there are 256 bricks in a pile arranged in the manner as in problem 14, how many bricks are there in the 3 rd row from the bottom of the pile ?

16. The sum of the first 4 terms of an A.P. is 26 and the sum of their squares is 214. Find the first 4 terms.

17. Insert three arithmetic means between -5 and 19.

18. Find the A .M. betwee.:

(a) -3 and 3 .

(b) $2-\sqrt{2}$ and $2+\sqrt{2}$

(c) log 3 and log 12.

5.4 Geometric Progression (G.P.)

A geometric progression is a sequence in which the ratio of each term to the one before it, is constant . The ratio is called the **common ratio** and is denoted by r. Some examples of geometric progressions are

(i) 3, 6, 12, 24, 48, ---

(ii) 2, -4, 8, -16, 32, ---

(iii) 2, 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, ---

The common ratio for (i) is 2, for (ii) is -2 , and for (iii) is $\frac{1}{2}$.

General term of a geometric progression

If we denote the first term of a geometric progression is denoted by a, then

$$u_1 = a, \qquad \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = --- = \frac{u_n}{u_{n-1}} = --- = r$$

Thus,

$$u_2 = u_1 r = ar$$

$$u_3 = u_2 r = ar^2$$

$$u_4 = u_3 r = ar^3 \quad \text{and so on.}$$

So , the general (or) standard form of a geometric progression is

a, ar, ar^2, ar^3, ---

and the general term (or) the n th term is given by

$$u_n = a r^{n-1}$$

Example 1. The fourth term of a G.P. is 9 and the ninth term is 2187 . Find the first 4 terms of the G.P.

Solution

$u_4 = 9$, $u_9 = 2187.$

Since $u_4 = ar^3$ and $u_9 = ar^8$,

$ar^3 = 9$ ---------- (1)

$ar^8 = 2187$

$$\therefore \frac{ar^8}{ar^3} = \frac{2187}{9}$$

$$r^5 = 243$$

$$r = 3$$

Substituting $r = 3$ in equation (1),

$$a(3)^3 = 9$$

$$a = \frac{1}{3}.$$

$\therefore$ The first 4 terms of the G.P . are $\frac{1}{3}$, 1,3, 9

Example 2. The fourth term of a G.P exceeds the third by $\frac{3}{44}$ and the third term exceeds the second term by $\frac{1}{22}$. Find the first term and the sixth term of the G.P.

Solution

$$u_4 - u_3 = \frac{3}{44}$$

$$u_3 - u_2 = \frac{1}{22}$$

Thus, $ar^3 - ar^2 = \frac{3}{44}$ and $ar^2 - ar = \frac{1}{22}$

$r(ar^2 - ar) = \frac{3}{44}$ ---------- (1)

$$r\left(\frac{1}{22}\right) = \frac{3}{44}$$

$$r = \frac{3}{2}$$

$$ar^2 - ar = \frac{1}{22}$$

$$a\left(\frac{3}{2}\right)^2 - a\left(\frac{3}{2}\right) = \frac{1}{22}$$

$$\frac{3}{4}a = \frac{1}{22}$$

$$a = \frac{2}{33}$$

$$u_6 = ar^5 = \frac{2}{33}\left(\frac{3}{2}\right)^5 = \frac{81}{176}$$

Example 3. Three consecutive terms of a G.P. are 3^{2x-1}, 9^x and 243 . Find the value of x. If 243 is the fifth term of the G.P., find the seventh term.

Solution

$3^{2x-1}, 9^x, 243$ is a G.P.

$$\therefore \quad \frac{9^x}{3^{2x-1}} = \frac{243}{9^x}$$

$$\frac{3^{2x}}{3^{2x-1}} = \frac{3^5}{3^{2x}}$$

$$3^{2x-(2x-1)} = 3^{5-2x}$$

$$3 = 3^{5-2x}$$

$$5 - 2x = 1$$

$$x = 2$$

$$ar^4 = 243$$

But $r = \dfrac{9^x}{3^{2x-1}} = \dfrac{9^2}{3^3} = \dfrac{81}{27} = 3$

$\therefore \quad u_7 = ar^6 = ar^4 . r^2 = 243\,(3)^2 = 2187$

Geometric Mean G.M.

In a finite geometric progression, the terms between the first term and the last term is called the **geometric means.**

For example, in the geometric progression ,3, 6, 12, 24,the geometric means are 6 and 12.

Exercise 5.4

1. Find (a) the common ratio, (b) the 10th term and (c) the nth term of each of the following G.P.

   (i) $4, 2, 1, \frac{1}{2}, \ldots$

   (ii) $-2, 4, -8, 16, \ldots$

   (iii) $5, 20, 80, 320, \ldots$

   (iv) $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$

   (v) $\frac{8}{9}, \frac{4}{3}, 2, 3, \ldots$

2. If 3, x, y, z, w and 3072 are consecutive terms of a G.P., find the values of x, y, z and w.

3. The second term of a G.P. is 64 and the fifth term is 27. Find the first 6 terms of the G.P.

4. Find the 10 th term of the G.P. a^5, a^4b, a^3b^2, a^2b^3, ...

   Which term of the G.P. is $\frac{b^{20}}{a^{15}}$?

5. The 4^{th} term of a G.P. is 3 and the sixth term is 147. Find the first 3 terms of the two possible geometric progressions.

6. The product of the first 3 terms of a G.P. is 1 and the product of the third, fourth and fifth terms is $11\frac{25}{64}$. Find the fifth term of the G.P.

7. Find two different values of x, so that $-\frac{3}{2}, x, -\frac{8}{27}$ will be a G.P.

8. Find which term of the G.P. $\frac{8}{9}, \frac{4}{3}\sqrt{\frac{2}{3}}, \frac{4}{3}, \ldots\ldots$ is $\sqrt{6}$.

9. If a, b, c, d is a G.P. show that $a^2 - b^2$, $b^2 - c^2$, $c^2 - d^2$ is also a G.P.

10. If a, b, c, d is a G.P., show that

(i) $$\frac{b+c}{c+d}=\frac{a+c}{b+d}$$

(ii) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$

11. If a, b, c is an A.P. and x, y, z is a G.P. show that $x^{b-c}y^{c-a}z^{a-b}=1$.

12. In a G.P. the product of three consecutive terms is 512. When 8 is added to the first term and 6 to the second, then the terms form an A.P. Find the terms of a G.P.

Sum of a geometric progression

Let S_n denote the sum of the first n terms of the G.P. a, ar, ar^2, ..., ar^{n-1} where a is the first term and r is the common ratio such that $r \neq 1$.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \qquad \text{------------- (1)}$$

When equation (1) is multiplied throughout by r, we have

$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n. \qquad \text{------------- (2)}$$

By subtracting equation (2) from equation (1),

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

If $r = 1$, then, $S_n = \underbrace{a + a + a + \dots + a}_{n \text{ times}}$

$\therefore S_n = na$

Example 4. Find the sum of the first 8 terms of the G.P. 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, ...

Solution

$a = 3, \qquad r = \frac{2}{3}, \qquad n = 8$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_8 = \frac{3(1-(\frac{2}{3})^8)}{1-\frac{2}{3}} = 8.65 \text{ (correct to 3 significant figures)}$$

Example 5. Given that $x + 1$, $x + 5$ and $2x + 4$ are three positive consecutive terms of a geometric progression calculate

(i) the value of x

(ii) the sum of the first 15 terms of the progression if $x + 1$ is the third term of the progression, giving your answer correct to 1 decimal place.

Solution

(i) $x + 1 , x + 5 , 2x + 4$ is a G.P.

$$\therefore r = \frac{x+5}{x+1} = \frac{2x+4}{x+5}$$

$$(x + 5)^2 = (x + 1)(2x + 4)$$

$$x^2 + 10x + 25 = 2x^2 + 6x + 4$$

$$x^2 - 4x - 21 = 0$$

$$(x - 7)(x + 3) = 0$$

$$x = 7 \quad \text{or} -3$$

Since the 3 terms of a G.P. are positive , $x = -3$ is not applicable

$$\therefore x = 7$$

(ii) $r = \frac{x+5}{x+1} = \frac{12}{8} = \frac{3}{2}$

$$u_3 = x + 1$$

$$ar^2 = x + 1$$

$$a(\frac{3}{2})^2 = 7 + 1$$

$$a = 8 \times \frac{4}{9} = \frac{32}{9}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{15} = \frac{\frac{32}{9}(1 - (\frac{3}{2})^{15})}{1 - \frac{3}{2}}$$

$$= 3106.8 \text{ (correct to 1 decimal place)}$$

Exercise 5.5

1. Find the sum of the first 10 terms for each of the following G.P.

   (i) $4, 2, 1, \frac{1}{2}, \dots$

   (ii) $-2, 4, -8, 16, \dots$

   (iii) $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \dots$

   (iv) $5, 20, 80, 320, \dots$

2. Show that $1 + \sqrt{2} + 2 + 2\sqrt{2} + \dots$ to 12 terms $= 63(\sqrt{2} + 1)$

3. Solve the equation

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} = x + 3 + \frac{x^{12}}{x-1}.$$

4. Find n, if

   (i) $3 + 3^2 + 3^3 + \dots + 3^n = 120$

   (ii) $2 + 2^2 + 2^3 + \dots + 2^n = 510$

5. The ratio of the sum of the first, second and third terms of a geometric progression to the sum of the third, fourth and fifth terms is 4 : 9. Find the tenth term of the progression if the sixth term is $15\frac{3}{16}$.

6. The sum of the fourth and sixth terms of a G.P. is 90 and the sum of the seventh and ninth terms is 2430. Find the sum of the first 17 terms of the G.P.

7. The numbers 28, x and y are in geometric progression. If the sum of these 3 terms is $21\frac{1}{7}$, find the possible values of x and of y.

8. The sum of the first 5 terms of a G.P. is 8 and the sum of the terms from the fourth to the eight inclusive is $15\frac{5}{8}$. Find the common ratio and the sixth term.

9. The length of the sides of a triangle form a G.P. If the shortest side is 9cm and the perimeter is 37 cm, find the length of the other two sides.

10. Find x if the numbers $x + 3$, $5x - 3$ and $7x + 3$ are three consecutive terms of a G.P. of positive terms. With this value of x and given that $x + 3$, $5x - 3$ and $7x + 3$ are the third, fourth and fifth terms of the G.P., find the sum of the first 8 terms of the progression.

11. Insert two geometric means between 2 and 128.

12. The ratio of two positive numbers is 9:1. If the sum of the arithmetic mean and positive geometric mean between the two numbers is 96, find the two numbers.

13. If the arithmetic mean between x and y is 15 and the geometric mean is 9, find x and y.

14. The sum of the first n terms of a certain sequence is given by $S_n = n^2 + 2n$. Find the first 3 terms of the sequence and express the n th term in terms of n.

5.5 Infinite Geometric Series

Consider the following geometric series

(a) $1 + 2 + 4 + 8 + \ldots$

(b) $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

If we keep on adding terms to the series indefinitely , we will get an **infinite series.** Thus an infinite series is the sum of an unlimited number of terms. In (a) , it is obvious that the more terms we take, the larger each term becomes and thus the sum will increase indefinitely . We say that the sum tends to infinity (represented by the symbol ∞). We cannot find a definite sum for this series and the series is said to be **divergent.**

Note: For example, if $y = \frac{1}{x}$, then y becomes infinitely large, or approaches infinity, as x approaches 0 from the positive side. An infinitely large negative quantity is denoted by $-\infty$ and an infinitely large positive quantity by $+\infty$.

In (b), $a = 2$, $r = \frac{1}{2}$

The sum of the first n terms of the series is given by

$$S_n = \frac{2\{1-(\frac{1}{2})^n\}}{1-\frac{1}{2}} = 4\{1-(\frac{1}{2})^n\} = 4-4.(\frac{1}{2})^n$$

From the above, we see that as n becomes larger i.e. when there are more and more terms in the series, the value of $(\frac{1}{2})^n$ becomes smaller and smaller until it becomes negligible. We say that as n tends to infinity, $(\frac{1}{2})^n$ tends to zero and S_n tends to 4.

Symbolically, we write as $n \to \infty$, $(\frac{1}{2})^n \to 0$ and $S_n \to 4$. We say that the **infinite** series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is **convergent** and converges to a sum of 4. **The value** 4 is called the **sum to infinity** of the series and we write

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 4$$

General Case

Consider the geometric progression $a, ar, ar^2, ar^3, \ldots$

The sum of the first n terms

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

Case 1: When r is numerically less than 1 (i.e. $-1 < r < 1$, written as $|r| < 1$),

$r^n \to 0$ as $n \to \infty$ and hence $\frac{ar^n}{1-r} \to 0$.

Thus S_n approaches a finite value $\frac{a}{1-r}$ as $n \to \infty$.

Case 2: (a) When $r > 1$, $r^n \to \infty$. Hence $\frac{ar^n}{1-r} \to -\infty$ when $a > 0$ and $\frac{ar^n}{1-r} \to \infty$ when $a < 0$.

Thus the sum to infinity cannot be found.

(b) When $r < -1$ and n is even, $r^n \to \infty$ as $n \to \infty$

When $r < -1$ and n is odd, $r^n \to -\infty$ as $n \to \infty$

Hence, $\frac{ar^n}{1-r}$ alternates between an increasing positive value and a decreasing negative value. It does not have a definite value. Thus the sum to infinity cannot be found.

When $r = -1$, the G.P. becomes $a, -a, a, -a, \ldots$

Hence, $\frac{ar^n}{1-r}$ alternates between the value of a and 0.

Thus the sum to infinity cannot be found either.

Case 3: When $r = 1$, the G.P. becomes a, a, a, ... and $S_n \to \infty$ as $n \to \infty$ Thus the sum to infinity cannot be found.

The sum to infinity of a G.P. can be found only when $|r| < 1$ and is given by the formula

$$S = \frac{a}{1-r}$$

Example 6. Determine whether the sum to infinity for each of the following geometric progressions exist and find the sum to infinity where they exist.

(i) $3, 0.3, 0.03, \ldots$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \frac{32}{27}, \ldots$

(iii) $\frac{7}{2}, 3, \frac{18}{7}, \frac{108}{49}, \ldots$

Solution

(i) $r = \frac{0.3}{3} = 0.1$

$\therefore\ |r| = 0.1 < 1$ and hence the sum to infinity exists.

$$S = \frac{a}{1-r} \quad \frac{3}{1-0.1} = \frac{10}{3}$$

(ii) $r = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3}$

$|r| = \frac{4}{3} > 1$ and hence the sum to infinity does not exist.

(iii) $r = 3 \times \frac{2}{7} = \frac{6}{7}$.

$|r| = \frac{6}{7} < 1$ and hence the sum to infinity exist.

$$S = \frac{a}{1-r} = \frac{\frac{7}{2}}{1-\frac{6}{7}} = \frac{49}{2}.$$

Example 7. A geometric progression is defined by $u_n = \frac{1}{3^n}$. Find S_n and the smallest value of n for which the sum of n terms and the sum to infinity differ by less than $\frac{1}{100}$.

Solution

$$u_n = \frac{1}{3^n}$$

$$u_1 = \frac{1}{3},\ u_2 = \frac{1}{9}$$

$$\therefore r = \frac{1}{9} \times \frac{3}{1} = \frac{1}{3}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{3}(1-(\frac{1}{3})^n)}{1-\frac{1}{3}} = \frac{1}{2}(1-\frac{1}{3^n})$$

Let S be the sum to infinity.

$$S = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

$$S - S_n < \frac{1}{100}$$

$$\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{3^n}) < \frac{1}{100}$$

$$\frac{1}{3^n} < \frac{1}{50}$$

$$3^n > 50$$

$$n \log 3 > \log 50$$

$$n > \frac{\log 50}{\log 3}$$

$$n > \frac{1.6990}{0.4771} = 3.562$$

Thus the smallest value of n is 4.

Exercise 5.6

1. Find the sum to infinity of each of the following series.

   (i) $2 + \frac{4}{3} + \frac{8}{9} + \ldots$ (ii) $2 + \frac{1}{2} + \frac{1}{\ } + \frac{1}{32} + \ldots$

   (iii) $3 - \frac{2}{3} + \frac{4}{27} - \ldots$ (iv) $81 - 27 + 9 - 3 + \ldots$

2. The third and sixth terms of a geometric progression are 9 and $2\frac{2}{3}$ respectively. Calculate the common ratio, the first term and the sum to infinity of the progression.

3. The sum of an infinite geometric progression is 12 and its first term is 3. Find the first 4 terms of the G.P.

4. In a G.P. the ratio of the sum of the first 3 terms to the sum to infinity of the G.P. is 19:27. Find the common ratio.

5. The second term of a G.P. is 2 and its sum to infinity is 9. Find the sum of the first 4 terms of the two possible geometric progressions.

6. The sum of the first three terms of a G.P. is 27 and the sum of the fourth, fifth and sixth terms is -1. Find the common ratio and the sum to infinity of the G.P.

7. Given that $x + 18$, $x + 4$ and $x - 8$ are the first three terms of a G.P., find the value of x. Hence, find

   (i) the common ratio (ii) the fifth term (iii) the sum to infinity.

8. Given that $2x - 14$, $x - 4$ and $\frac{1}{2}x$ are successive terms of a sequence.

   (a) find the value of x when the sequence is

   (i) an A.P. (ii) a G.P.

   (b) If $2x - 14$ is the 3rd term of a G.P. with infinite terms, find

   (i) the common ratio

   (ii) the sum to infinity.

9. Given that 8, p and q are three consecutive terms of an A.P. while p, q and 36 are three consecutive terms of a G.P., find the possible values of p and q.

10. Find the smallest value of n for which the sum to n terms and the sum to infinity of a G.P. $1, \frac{1}{5}, \frac{1}{25}, \ldots$ differ by less than $\frac{1}{1000}$.

SUMMARY

1. The general form of an A.P. is a, a + d, a + 2d, ...
   where a = first term, d = common difference.
2. The n th term of an A.P. is given by $u_n = a + (n - 1)d$.
3. The sum of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}\{a + \ell\}, \qquad \text{where } \ell = \text{last term or n th term.}$$

(or) $$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

4. A.M. between a and b = $\dfrac{a+b}{2}$
5. The general form of a G.P. is a, ar, ar^2, ...
   where a = first term and r = common ratio.
6. The n th term of a G.P. is $u_n = ar^{n-1}$.
7. The sum of the first n terms of a G.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r-1}, \quad r \neq 1$$

8. G.M. between a and b = $\sqrt{ab}$
9. The sum to infinity of a G.P. exists only when $|r| < 1$ and is given by

$$S = \frac{a}{1-r}.$$

CHAPTER 6
Matrices

Matrix was introduced in 1850 by the English Mathematician James Joseph Sylvest·r. It did not take long before mathematicians realised that matrices are a convenient device for extending the common notions of numbers. Sir William Rowan Hamilton and Arthur Cayky made further contributions to the subject.

The theory of matrices is, in the main, a part of algebra. But it becomes clear that matrices possessed a utility that extended beyond the domain of algebra and into other regions of mathematics. It was found that they were the means necessary for expressing many ideas of applied mathematics. Today, matrices are used in mathematics and other sciences.

6.1 Matrices

Consider two simultaneous linear equations in two unknown x and y,

$$3x + y = 5$$
$$4x - y = 2$$

The coefficients of x and y could be put down in the form of a rectangular array without altering their relative positions in the equations, thus:

$$\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}$$

Such an array is called a matrix.

Definition

A matrix (plural : matrices) is a rectangular array of numbers arranged in rows and columns, the array being enclosed in round brackets.

The rows of a matrix are the arrays of numbers that go across the page. The columns are those that go down the page.

The order of a matrix is given by the number of rows followed by the number of columns, and it is denoted by $m \times n$, if the matrix has m rows and n columns.

For example :

(i) $\begin{pmatrix} 2 & 4 & 7 \\ 1 & 6 & 2 \end{pmatrix}$ is a 2×3 matrix, since it has 2 rows and 3 columns.

(ii) $\begin{pmatrix} 5 & 2 & 7 \\ -1 & 0 & 3 \\ 3 & 4 & 1 \end{pmatrix}$ is a 3×3 matrix, since it has 3 rows and 3 columns.

In general, a matrix of order 2×3 can be written

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

and a matrix of order 3×3 can be written

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Here a_{11}, a_{12}, a_{13} , ... are called the elements or entries of the matrix. Notice that a_{12} is the element in the 1st row and the 2nd column and a_{32} is the element in the 3rd row and the 2nd column of the matrix. Thus a_{ij} will denote the element in the i^{th} row and j^{th} column of the matrix.

Capital letters are usually used to represent matrices, e.g. three matrices could be represented as A, B, C.

Square matrix : When the numbers of rows in a matrix is the same as the number of columns, the matrix is called a square matrix. The following are square matrices.

$A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ is a square matrix of order 2.

$B = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 2 & 1 \\ 2 & -1 & 5 \end{pmatrix}$ is a square matrix of order 3.

In practical applications, the entries or elements of matrices come from the world in which we live and have physical or economic or social meanings.

Information is often presented in matrix form in everyday life, in economics, in mathematics and sciences. Here are some examples.

Example 1. This example comes from the weather report for a day.

	highest temperature	lowest temperature	rainfall in inches
Yangon	94	63	.12
Mandalay	96	73	.00
Mawlamyine	81	60	.24

The matrix for this example is

$$\begin{pmatrix} 94 & 63 & .12 \\ 96 & 73 & .00 \\ 81 & 60 & .24 \end{pmatrix}$$

Example 2. A road map has the following mileage chart.

	Yangon	Bago	Nyaunglebin
Yangon	0	50	98
Bago	50	0	48
Nyaunglebin	98	48	0

The matrix

$$\begin{pmatrix} 0 & 50 & 98 \\ 50 & 0 & 48 \\ 98 & 48 & 0 \end{pmatrix}$$ represents the above informations.

Exercise 6.1

1. Answer questions (a) to (e) for the matrix

$$\begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 2 & 6 & 9 & 12 \end{pmatrix}$$

(a) State (i) the number of rows (ii) the number of columns.
(b) List the elements in the second row.
(c) List the elements in the third column.
(d) Write down the entry in :
 (i) The first row and first column.
 (ii) The third row and third column.
(e) State the rows and columns which describe the position of these entries.
 (i) 4 (ii) 9 (iii) 6 (iv) 11 (v) 2
 (vi) 5

2. For each of the following matrices, state the order of matrix and the entry in the second row and first column.

(a) $\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 7 \\ 3 & 0 \\ 5 & 9 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 5 & 6 & 7 \end{pmatrix}$

3. In each of the following systems of equations, write down the matrix of coefficients of the variable x and y.

(a) $2x + 3y = 4$ (b) $x - 3y = 2$ (c) $y = 3$
$4x + 5y = 3$ $2x + y = 3$ $x + 3y = 5$

4. Write down examples of matrices with numerical elements arranged in

(a) 1 row and 3 columns (b) 2 rows and 3 columns
(c) 3 rows and 2 columns (d) 5 rows and 2 columns.

6.2 Equality of Matrices

The position of an element in a matrix is fundamental importance. If different elements are interchanged, the matrix itself is changed. For example,

$$\begin{pmatrix} 2 & 1 \\ 5 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 \\ 1 & 7 \end{pmatrix}, \quad \text{are all different.}$$

Two matrices are identical if and only if each element of one is equal to the corresponding element of the other.

For instance, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

if and only if

$a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{13} = b_{13}$, $a_{21} = b_{21}$, $a_{22} = b_{22}$, $a_{23} = b_{23}$

Further, for example,

$$\begin{pmatrix} x & 4 \\ 3 & y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

$\Leftrightarrow x = 1$ and $y = 2$.

Two matrices cannot be equal unless they have the same number of rows (m, say) and the same number of columns (n, say). Thus

$$\begin{pmatrix} 2 & 0 \\ -3 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Thus we have found that equality of matrices means identical matrices.

Defintition

Two matrices are said to be equal if

(i) they are of the same order, and (ii) their corresponding entries are equal.

Example 1. Find x and y in each of the following .

Solution

(a) $$\begin{pmatrix} x & 2y \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 3 \end{pmatrix}$$

$x = 1$ $\qquad$ $2y = 8$

Therefore $x = 1,$ $\qquad$ $y = 4$

(b) $$\begin{pmatrix} x + y \\ x - y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + & \text{..................(i)} \\ x - y = 6 & \text{..................(ii)} \end{cases}$$

Adding (i) and (ii), we get $2x = 10$

So $x = 5$

and hence $y = -1$

Exercise 6.2

1. List any equalities for pairs of the following matrices.

$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ $\qquad$ $C = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$D = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\qquad$ $E = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\qquad$ $F = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$G = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad$ $H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad$ $J = \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$

$K = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ $\qquad$ $L = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad$ $M = \begin{pmatrix} 1 & 2 \end{pmatrix}$

Note : A, B, C and M are called row matrices and D, E, F and G are called column matrices.

2. What is the order of each matrix in question 1 ?
3. Find x and y in each of the following.

(a) $\begin{pmatrix} 3x & -y \end{pmatrix} = \begin{pmatrix} 12 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} x+3 \\ 4-y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$

(d) $\begin{pmatrix} x^2 & y^2 \\ y^3 & x^3 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -27 & 8 \end{pmatrix}$

(e) $\begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & -8 \end{pmatrix}$

6.3 Transpose of a Matrix

From a given matrix A, a new matrix can be formed by writing row 1 as column 1, row 2 as column 2 and so on. This new matrix is called the transpose of A and is denoted by A' (read as A transpose).

For example,

if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Definition

Let A be a matrix of order m × n. A matrix of order n × m whose rows are columns and whose columns are rows of A is called the transpose of A and is denoted by A'.

Exercise 6.3

1. Write down the transpose of each of the following matrices, and state the order of each transpose:

(a) $\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & e \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$

2. Write down the total number of the entries in each matrix in problem 1. Do you see a quick way to find the answers?

3. Let $P = \begin{pmatrix} x & 5 \\ -3 & y \end{pmatrix}$ and $Q = \begin{pmatrix} 9 & -3 \\ 5 & -7 \end{pmatrix}$

   Find x and y, given that $P = Q'$.

4. If $B = \begin{pmatrix} 2 & -3 & 4 & -1 \\ -9 & 1 & 5 & 0 \end{pmatrix}$, find $(B')'$

6.4 Addition of Matrices

With a view to defining addtition on a set of matrices, we will study the following example.

Example Mg Mg and Kyaw Kyaw, who are close rivals in the mathematics class, compare their marks in Mathematics and Science at the end of the second term.

First test

	Mg Mg	Kyaw Kyaw
Mathematics	82	78
Science	68	72

Second test

	Mg Mg	Kyaw Kyaw
Mathematics	75	80
Science	70	78

Total

	Mg Mg	Kyaw Kyaw
Mathematics	82+75=157	78+80=158
Science	68+70=138	72+78=150

Setting out this information in matrix form, it is reasonable to write :

$$\begin{pmatrix} 82 & 78 \\ 68 & 72 \end{pmatrix} + \begin{pmatrix} 75 & 80 \\ 70 & 78 \end{pmatrix} = \begin{pmatrix} 82+75 & 78+80 \\ 68+70 & 72+78 \end{pmatrix} = \begin{pmatrix} 157 & 158 \\ 138 & 150 \end{pmatrix}$$

This method of combining matrices is called addition of matrices.

The above example shows that the addition of matrices is simple but may only be carried out when the matrices are of the same order. Corresponding elements are added.

Definition

If A and B are two matrices of the same order, the sum of A and B, denoted by $A + B$, is the matrix obtained by adding the entries of A and the corresponding entries of B.

Two important facts follow from the definition :

(1) The matrix $A + B$ will be of the same order as each of A and B.

(2) It is not possible to add two matrices of different orders.

Example 1.

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & 0 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 1 & 1 \\ 3 & 5 & -5 \end{pmatrix} = \begin{pmatrix} 3-2 & 2+1 & -1+1 \\ 2+3 & 0+5 & 5-5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 0 \\ 5 & 5 & 0 \end{pmatrix}$$

Remark : Intermediate steps may be omitted after a little practice.

Example 2.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -2a & b \\ -c & 2d \end{pmatrix} = \begin{pmatrix} -a & 2b \\ 0 & 3d \end{pmatrix}$$

Definition

A matrix whose elements are all zero is called a zero matrix. It is denoted by O.

Example 1.

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the 2×3 zero matrix.

Example 2. Given that $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Show that $O + A = A + O = A$.

Solution

$$O + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

$$A + O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

Therefore $\quad O + A = A + O = A$

Note : The zero matrix is the identity element for addition of matrices.

Example 3. Given that $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix}$, find A + B and B + A and hence show that $A + B = B + A = O$.

Solution

$$A + B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$B + A = \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Hence $\quad A + B = B + A = O$

The example provides us with some ideas about the negative of a matrix.

Each entry in B is the negative of the corresponding entry in A. For this reason, B is called the negative of A and is written – A.

Definition

If A is a matrix, then negative of A, written – A, is the matrix in which each entry is the negative of the corresponding entry in A.

Example.

If $B = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 1 & -5 \end{pmatrix}$, then

$$-B = \begin{pmatrix} -1 & -(-2) & -3 \\ -(-4) & -1 & -(-5) \end{pmatrix} = \begin{pmatrix} -1 & 2 & -3 \\ 4 & -1 & 5 \end{pmatrix}$$

Note :

Since $A + (-A) = (-A) + A = O$, we call $-A$ the additive inverse of A, so that $-(-A) = A$.

Exercise 6.4

1. Perform the following additions and subtractions **where they are possible**, where they are not write "not possible".

(i) $(1) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(v) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ (vi) $\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$

(vii) $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (viii) $(9 \quad 6) + (7 \quad 8)$

2. (a) $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$

Find the matrices A + B and B + A.

(b) Is it true that A + B = B + A ? What law for matrix addition does this result suggest?

3. $A = \begin{pmatrix} 5 & 5 \\ 4 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 7 \\ 4 & -3 \end{pmatrix}$

(a) Find the following matrices.

(i) A + B (ii) B + C (iii) (A + B) + C

(iv) A + (B + C)

(b) Is it true that (A + B) + C = A + (B + C)?

What law for addition of matrices does this suggest?

4. Write down the negative of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -8 \\ 4 & -7 \end{pmatrix}$

(d) $\begin{pmatrix} -4 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix}$

5. In each of the following cases : find the matrix A which satisfies the given relationship.

(i) $A + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ (ii) $A + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$

6. Solve each of the following equation for the 2×2 matrix X.

(a) $X + \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} + X = \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$

6.5 Multiplication of Matrix by a Real number

Let $X = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$. From the definition of addition of matrices.

$$X + X = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix} = \begin{pmatrix} 2\times3 & 2\times4 \\ 2\times5 & 2\times6 \end{pmatrix}$$

and

$$X + X + X = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 15 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} 3\times3 & 3\times4 \\ 3\times5 & 3\times6 \end{pmatrix}$$

If we now write

$$2\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 2\times3 & 2\times4 \\ 2\times5 & 2\times6 \end{pmatrix}$$

and

$$3\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 3\times3 & 3\times4 \\ 3\times5 & 3\times6 \end{pmatrix}$$

it is reasonable to denote X + X by 2X and X + X + X by 3X. Extending this idea, we make the following definition.

Definition

If k is a real number and A is a matrix, then k A is the matrix, obtained by multiplying each entry of A by k. This operation of multiplying A by k is called scalar multiplication.

Note : (Subtraction of Matrices)

If A and B are two matrices of the same order, $A - B = A + (-1)B$

Example 1. Give that $A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$

find in their simplest forms the matrices.

(a) 2A (b)(−1)A (c) 3A − 2B

Solution

(a) $2A = 2\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2\times2 & 2\times1 \\ 2\times4 & 2\times3 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 8 & 6 \end{pmatrix}$

(b) $(-1)A = (-1)\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -4 & -3 \end{pmatrix}$

Note: This shows that (-1) A is the negative of A; that is (-1) A = −A. This result is true for any matrix A.

(c) $3A = 3\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 12 & 9 \end{pmatrix}$

$2B = 2\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 10 \\ 0 & 4 \end{pmatrix}$

$3A - 2B = \begin{pmatrix} 6 & 3 \\ 12 & 9 \end{pmatrix} - \begin{pmatrix} 2 & 10 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6-2 & 3-10 \\ 12-0 & 9-4 \end{pmatrix} = \begin{pmatrix} 4 & -7 \\ 12 & 5 \end{pmatrix}$

Note : From the meaning of k A, it readily follows that

(i) 0A = O and (ii) kO = O, when O is a zero matrix of suitable order.

Example 2. Solve $5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 3X = 4\begin{pmatrix} -4 & 7 \\ 3 & 8 \end{pmatrix}$

for the 2×2 matrix X.

Solution

$$5\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} - 3X = 4\begin{pmatrix}-4 & 7\\ 3 & 8\end{pmatrix}$$

$$\begin{pmatrix}5 & 10\\ 15 & 20\end{pmatrix} - 3X = \begin{pmatrix}-16 & 28\\ 12 & 32\end{pmatrix}$$

$$\begin{pmatrix}5 & 10\\ 15 & 20\end{pmatrix} + \begin{pmatrix}-5 & -10\\ -15 & -20\end{pmatrix} - 3X = \begin{pmatrix}-16 & 28\\ 12 & 32\end{pmatrix} + \begin{pmatrix}-5 & -10\\ -15 & -20\end{pmatrix}$$

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} - 3X = \begin{pmatrix}-21 & 18\\ -3 & 12\end{pmatrix}$$

$$O - 3X = \begin{pmatrix}-21 & 18\\ -3 & 12\end{pmatrix}$$

$$-3X = \begin{pmatrix}-21 & 18\\ -3 & 12\end{pmatrix}$$

$$(-\frac{1}{3})(-3X) = -\frac{1}{3}\begin{pmatrix}-21 & 18\\ -3 & 12\end{pmatrix}$$

$$X = \begin{pmatrix}7 & -6\\ 1 & -4\end{pmatrix}$$

Exercise 6.5

1. If $A = \begin{pmatrix}4 & 4\\ 2 & 7\end{pmatrix}$, $B = \begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$, $C = \begin{pmatrix}1 & 4\\ 3 & -5\end{pmatrix}$,

   find (i) $A + B$. (ii) $A + 2B$.

   (iii) $A + B - C$ (iv) $2A - 2C + 3B$

2. Solve the following equations:

   (i) $a\begin{pmatrix}2\\ 2\end{pmatrix} + b\begin{pmatrix}2\\ -2\end{pmatrix} = \begin{pmatrix}0\\ 8\end{pmatrix}$ (ii) $3\begin{pmatrix}2x\\ y\end{pmatrix} + 3\begin{pmatrix}x\\ 3y\end{pmatrix} = \begin{pmatrix}18\\ 36\end{pmatrix}$

   (iii) $2\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} - \begin{pmatrix}3 & 5\\ c & 6\end{pmatrix} = \begin{pmatrix}a & b\\ 7 & d\end{pmatrix}$ (iv) $\begin{pmatrix}2\\ -5\end{pmatrix} + 3\begin{pmatrix}7\\ a\end{pmatrix} = \begin{pmatrix}2a\\ b\end{pmatrix}$

(v) $2\begin{pmatrix}5 & 3 & 2\\ 1 & 6 & 3\end{pmatrix}+\begin{pmatrix}a & b & c\\ -2 & -4 & 5\end{pmatrix}=\begin{pmatrix}9 & 12 & 6\\ d & e & f\end{pmatrix}$

3. If $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $C=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$, $D=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$,

find an expression in terms of A, B, C, D for the matrix $\begin{pmatrix}3 & 6\\ 7 & 9\end{pmatrix}$.

4. Find the matrix A in each of the following.

(i) $3A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$ (ii) $A+\begin{pmatrix}3\\ 6\end{pmatrix}=4A$

(iii) $\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}-A=\begin{pmatrix}1 & 7\\ 4 & 5\\ 3 & -4\end{pmatrix}$

5. Solve each of the following equations for the 2×2 matrix X.

(a) $3X=\begin{pmatrix}6 & -3\\ 12 & 9\end{pmatrix}$ (b) $2X-\begin{pmatrix}3 & 1\\ 4 & 2\end{pmatrix}=\begin{pmatrix}9 & 5\\ 2 & 8\end{pmatrix}$

(c) $4X-\begin{pmatrix}3 & 1\\ 4 & 7\end{pmatrix}=\begin{pmatrix}5 & 3\\ 0 & 13\end{pmatrix}$

6. Find the matrix X in each of the following.

(a) $2\begin{pmatrix}1 & -1 & 3\\ 2 & -7 & 5\end{pmatrix}+X=3\begin{pmatrix}1 & 2 & -4\\ 3 & -5 & 1\end{pmatrix}$

(b) $5\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}-3X=4\begin{pmatrix}-4 & 7\\ 3 & 8\end{pmatrix}$

7. Given that $A=\begin{pmatrix}2 & -2\\ 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}5 & a\\ c & 4\end{pmatrix}$ and $C=\begin{pmatrix}b & 6\\ 4 & d\end{pmatrix}$,

find the values of a, b, c and d when

(i) $2A+B=C$ (ii) $3A-2B=4C$.

6.6 Multiplication of Matrices

Can we multiply one matrix by another matrix? The following illustration will suggest an answer to this question. The student will need to observe some cases in studying this section as the multiplication is more exciting than the process of addition.

Table 1 shows the purchases of food made by a housewife in two consecutive months and Table 2 gives the cost of the food per viss.

Table 1

Purchase (viss)	Onions	Potatoes
First month	3	1
Second month	2	2

Table 2

Food	Cost in kyats per viss
Onions	5
Potatoes	6

The total cost of onions for the first month is $3 \times 5 = 15$ kyats.
The total cost of potatoes for the first month is $1 \times 6 = 6$ kyats.
The total cost of food for the first month is $3 \times 5 + 1 \times 6 = 15 + 6 = 21$ kyats.

Setting out the information in Table 1 and 2 in matrix form, the calculation may be shown as follows:

(i) $\begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = (3 \times 5 + 1 \times 6) = (15 + 6) = (21)$

In fact, we have multiplied each entry in the 1×2 row matrix by the corresponding entry in the 2×1 column matrix, and found the sum of these products as a 1×1 cost matrix.

The total cost for the second month is given by

(ii) $\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = (2 \times 5 + 2 \times 6) = (10 + 12) = (22)$

giving the cost as 22 kyats.

We can show the cost for both months as follows:

(iii) $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3\times5 + 1\times6 \\ 2\times5 + 2\times6 \end{pmatrix} = \begin{pmatrix} 15 + 6 \\ 10 + 12 \end{pmatrix} = \begin{pmatrix} 21 \\ 22 \end{pmatrix}$

This method of combining matrices is called multiplication of matrices.

The rule is "multiply each entry of a row in the first matrix by the corresponding entry of the column in the second matrix and then add the products to give the 2×1 matrix."

Multiplication of an m×p matrix by a p×1 matrix

Consider the two linear expressions:

$$\left.\begin{matrix} ax + by \\ \\ cx + dy \end{matrix}\right\} \qquad \text{.........................(1)}$$

By the above rule,(1) can be obtained from the product of the following 2×2 matrix and 2 × 1 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{............................(2)}$$

Therefore

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{.....................(3)}$$

Thus the system of linear equations

$$ax + by = p$$
$$cx + dy = q$$

can be written in matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Example 1. Find the product $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solution $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x-5y+6z \end{pmatrix}$

Example 2. Perform the matrix multiplication $(4 \quad 3 \quad 2) \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$

Solution

$$(4 \quad 3 \quad 2) \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = (4\times3 + 3\times1 + 2(-5)) = (12+3-10) = (5)$$

Multiplication of an m×p matrix by a p×n matrix

To illustrate a further extension of matrix multiplication, consider the mappings

$f:(x, y) \longmapsto (x', y')$ and $g:(x', y') \longmapsto (x'', y'')$ defined by

$$\left.\begin{aligned} x' &= tx + uy \\ y' &= rx + sy \end{aligned}\right\} \text{ and } \left.\begin{aligned} x'' &= ax' + by' \\ y'' &= cx' + dy' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

On substitution for x' and y'

$$\left.\begin{aligned} x'' &= a(tx + uy) + b(rx + sy) \\ y'' &= c(tx + uy) + d(rx + sy) \end{aligned}\right\}$$

$$\left.\begin{aligned} x'' &= (at + br)x + (au + bs)y \\ y'' &= (ct + dr)x + (cu + ds)y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

In matrix form, (2) can be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} at+br & au+bs \\ ct+dr & cu+ds \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Hence the mapping f followed by $g:(x, y) \longmapsto (x'', y'')$.

Expressing each pair of equations in (1) in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t & u \\ r & t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

from which, on substitution for $\begin{pmatrix} x' \\ y' \end{pmatrix}$ in the second pair,

we get $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & u \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ ....(4)

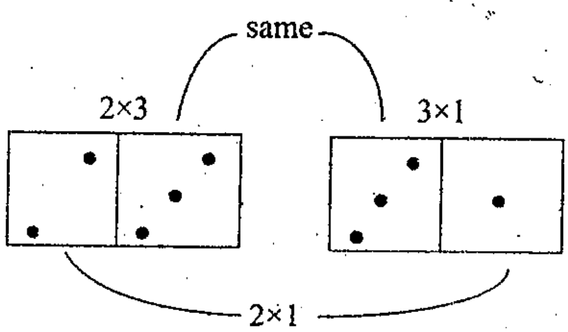
(3) and (4) show that the result of the mapping f followed by the mapping g could have been written down the following definition of multiplication of matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & u \\ r & s \end{pmatrix} = \begin{pmatrix} at+br & au+bs \\ ct+dr & cu+ds \end{pmatrix}$$

A little thouhgt will show that the "row in column" rule for multiplication of matrices requires that the number of columns in the left-hand matrix is the same as the number of rows in the right-hand matrix. Hence it is only possible to multiply on m × p matrix by a q×n matrix if q = p, and the product matrix will be of order m×n. The two matrices are then said to be conformable for multiplication.

Note : In checking whether or not a product exists and also in working out the order of the product matrix, a comparison with matching dominoes may be helpful as shown in the figure.

$$\begin{pmatrix} 3 & 2 & 1 \\ 5 & -3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+2y+z \\ 5x-3y+7z \end{pmatrix}$$



General definition of a matrix product

The product of an m×p matrix A and a p×n matrix B is the m×n matrix AB whose entry in the i^{th} row and j^{th} column is the sum of the products of corresponding entries in the i^{th} row of A and the j^{th} column of B.

Example 1. Given $P = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix}$, find PQ and QP.

Solution

$$PQ = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4+4 & 5+0 \\ 12+2 & 15+0 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 14 & 15 \end{pmatrix}$$

$$QP = \begin{pmatrix} 4 & 5 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4+15 & 8+5 \\ 2+0 & 4+0 \end{pmatrix} = \begin{pmatrix} 19 & 13 \\ 2 & 4 \end{pmatrix}$$

Notice that $PQ \neq QP$, so multiplication of matrices is not commutative. To avoid ambiguity in the multiplication of matrices, PQ may be described as P post multiplied by Q or Q pre-multiplied by P.

Example 2. If $A = (3 \quad 5)$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ which of the products

AB, BA are possible? Simplify those products that exist.

Solution

A   B

same

1×2 2×2 AB exists

1×2

$$AB = (3 \quad 5)\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= (6\text{-}5 \quad -3+0)$$

$$= (1 \quad -3)$$

B   A

different   BA does not exist.

2×2 1×2

Example 3. Given $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$, find a system of equations in x and y. Hence find x and y.

Solution $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3x+y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$$

Therefore $3x + y = 9$ ....(1)

$2x - y = 1$ ....(2)

(1) and (2) give $5x = 10$

$x = 2$

From (1) $y = 3$

Example 4. Find the products: $\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7+0 & 0+5 \\ 6+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}7 & 5\\6 & 4\end{pmatrix} = \begin{pmatrix}7+0 & 5+0\\0+6 & 0+4\end{pmatrix} = \begin{pmatrix}7 & 5\\6 & 4\end{pmatrix}$$

Note : The 2×2 matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is called the **unit matrix of order 2.** and denoted by I. It behaves like unity in the real system.
If A is a 2×2 matrix, then $IA = AI = A$ (see the above example.)

Example 5. If $A = \begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}$ and $B = \begin{pmatrix}2 & 2\\1 & 1\end{pmatrix}$, find AB and BA.

Solution

$$AB = \begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix}\begin{pmatrix}2 & 2\\1 & 1\end{pmatrix} = \begin{pmatrix}2+(-2) & 2+(-2)\\-6+6 & -6+6\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} = O$$

$$BA = \begin{pmatrix}2 & 2\\1 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\-3 & 6\end{pmatrix} = \begin{pmatrix}2+(-6) & -4+12\\1+(-3) & -2+6\end{pmatrix} = \begin{pmatrix}-4 & 8\\-2 & 4\end{pmatrix}$$

Note (1) : $AB = O$ does not necessarily mean that $A = O$ or $B = O$

Note (2) : Powers of a square matrix A are defined as follows :
$A^2 = AA$, $A^3 = AA^2$, $A^4 = AA^3$ and so on.

Example 6. If $A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$, find p, q such that $A^2 = pA + qI$.

Solution

$$A^2 = pA + qI$$

$$\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} = p\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} + q\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1+6 & 2+8\\3+12 & 6+16\end{pmatrix} = \begin{pmatrix}p & 2p\\3p & 4p\end{pmatrix} + \begin{pmatrix}q & 0\\0 & q\end{pmatrix}$$

$$\begin{pmatrix}7 & 10\\15 & 22\end{pmatrix} = \begin{pmatrix}p+q & 2p\\3p & 4p+q\end{pmatrix}$$

Hence
$$\left.\begin{aligned}p+q &= 7\\2p &= 10\\3p &= 15\\4p+q &= 22\end{aligned}\right\}$$ from which $p = 5$, $q = 2$.

Exercise 6.6

1. Find the following matrix products.

(a) $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} 5 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ (e) $\begin{pmatrix} 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$

2. Obtain the matrix products of the following where possible.

(a) $\begin{pmatrix} 1 \\ 7 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 7 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 3 & 1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & 3 \\ 1 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 7 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 5 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ (h) $\begin{pmatrix} \cos x & \sin x \end{pmatrix}\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$

3. In each of the following, find a system of equations in x and y. Hence find x and y.

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} x & y \\ y & x \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

4. Find the values of a and b if $\begin{pmatrix} a & 2a \\ 2b & b \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 33 \\ 50 \end{pmatrix}$.

5. If $A = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix}$, find (i) AB (ii) BA

(iii) the value of k if $AB = BA$.

6. Given that $2\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

find the values of a, b, c and d.

7. The matrices $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ are such that

$AB = A + B$. Find the values of a, b and c.

Example 1. $A = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$. Find A^2, A^3 and A^4 and hence deduce a formula for A^n, where n is a positive integer.

Solution

$$A^2 = AA = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^2+0 & k+k \\ 0+0 & 0+k^2 \end{pmatrix} = \begin{pmatrix} k^2 & 2k \\ 0 & k^2 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} k^2 & 2k \\ 0 & k^2 \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^3 & 3k^2 \\ 0 & k^3 \end{pmatrix}$$

$$A^4 = A^3A = \begin{pmatrix} k^3 & 3k^2 \\ 0 & k^3 \end{pmatrix}\begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k^4 & 4k^3 \\ 0 & k^4 \end{pmatrix}$$

Hence $A^n = \begin{pmatrix} k^n & nk^{n-1} \\ 0 & k^n \end{pmatrix}$, where n is a positive integer.

Example 2. Find the two matrices of the form $X = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}$ such that $X^2 = I$.

Solution

$$X^2 = XX = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}\begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x^2 & x+y \\ 0 & y^2 \end{pmatrix}$$

$$X^2 = I$$

$$\begin{pmatrix} x^2 & x+y \\ 0 & y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore\ x^2 = 1, \qquad x + y = 0, \qquad y^2 = 1$

$\therefore\ x = \pm 1, \qquad\qquad y = \pm 1$

For $x + y = 0$, $x = 1$, $y = -1$ or $x = -1, y = 1$

Thus the required two matrices are $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

Exercise 6.7

1. If $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$,

   find in simplest form :

   (a) (AB) C (b) A (BC) (c) (CB) A (d) C (BA)

   What law appears to hold?

2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

   (a) is $A + (B + C) = (A + B) + C$? (b) is $A (BC) = (AB) C$?

   (c) is $A + B = B + A$? (d) is $AB = BA$?

   (e) is $A (B + C) = AB + AC$? Can you give the name of this law.

   (f) is $A + (BC) = (A + B) (A + C)$?

3. $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$, find :

   (a) $A + B$ (b) $A - B$ (c) $(A + B)(A - B)$

   (d) A^2 (e) B^2

   Is it true that $(A + B)(A - B) = A^2 - B^2$?

4. For the matrices A and B given in problem (3), find (a) $(A+B)^2$ (b) $A^2 + 2AB + B^2$. Is it true that $(A + B)^2 = A^2 + 2AB + B^2$?

5. $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, verify that $A^2 - 2A + I = O$

where I is the unit matrix of order 2.

Show that the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ satisfies the equation $A^2 - 4A - 5I = O$.

6. Given that $D = \begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$ and that $D^2 - 3D - kI = O$, find the value of k.

7. Evaluate $(2A - B)C$ where $A = \begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 2 \\ -1 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

6.7 The Inverse of a Square Matrix of Order 2

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pre-multiplying A by I, $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

Post-multiplying A by I, $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

Therefore $IA = AI = A$

For this reason, the unit 2×2 matrix I is called the identity matrix for multiplying of 2×2 matrices. Note that A commutes with I, i.e. $IA = AI$.

Consider matrices $P = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $Q = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

Pre-multiplying Q by P, $PQ = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Post-multiplying Q by P, $QP = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Therefore $PQ = QP = I$

For this reason Q is called the multiplicative inverse of P and is denoted by P^{-1}. We can also say that P is the multiplicative inverse of Q and is therefore denoted by Q^{-1}.

It is customary to use the phrase "inverse of a matrix" to refer to its multiplicative inverse, since its additive inverse is usually called its negative.

Definition

If A and B are square matrices of the same order such that AB = BA = I, then B is an inverse of A and A is an inverse of B.

It can be shown that if these inverses exist, then they are unique; we can talk about the inverse of A or the inverse of B.

Example.

If $A = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$, show that A and B are inverses of each other.

Solution

We have to show that AB = I = BA.

$$AB = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$BA = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Since AB = I = BA, A and B are inverses of each other.

Note : From the above example we notice that

(i) the difference of the "cross-product" of the entries was always 1.

For example, from $\begin{pmatrix} 9 & -5 \\ -7 & 4 \end{pmatrix}$, $9 \times 4 - (-5) \times (-7) = 36 - 35 = 1$

(Note the order – the main diagonal product first)

(ii) the inverse matrix could be found by interchanging the entries in the main diagonal, and changing the signs of the entries in the other diagonal.

Exercise 6.8

In questions 1 to 5, show that each matrix is the inverse of the other.

1. $\begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}$

2. $\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -3 \\ -4 & 7 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

Study the patterns in the entries of the pair of matrices in question 1 to 4. Use this pattern to write down the inverse of each of the matrices in question 5 to 7 . Check by multiplication.

5. $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ 6. $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

7. $\begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$

8. Using the definition of inverse of matrix, find the inverse of each of the following matrices.

(a) $\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}$

9. Find the inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

10. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that I is its own inverse, i.e. $I \; I = I$.

11. $M = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$. Find M^{-1}, as in question 6.

Investigate whether or not the squares of M and M^{-1} are also inverses of each other.

12. $A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{pmatrix}$. Show that $AB = I = BA$ and so $B = A^{-1}$.

6.8 More about Inverse of Square Matrices of Order 2

Does every 2×2 matrix have an inverse?

To answer this question, consider the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Pre-multiplying A by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $\left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right] \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Similarly, post-multiplying A by $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we obtain

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows that if $ad - bc \neq 0$, the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has inverse

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$ad - bc$ i.e. the main diagonal product minus the other diagonal product is called the determinant of the matrix A and is written det A. If det A = 0, A does not have an inverse, and is called a **singular matrix.** If det $A \neq 0$, then A is said to be **non-singular.**

Example 1. Given that the value of the determinant of the matrix $\begin{pmatrix} 2a & -4 \\ -1 & 5 \end{pmatrix}$ is 16,

find the value of a. Hence, write down the inverse of the matrix.

Solution $\det\begin{pmatrix}2a & -4\\ -1 & 5\end{pmatrix} = 10a - 4 = 16$

$$10a = 20$$
$$a = 2$$

The given matrix is $\begin{pmatrix}2\times 2 & -4\\ -1 & 5\end{pmatrix} = \begin{pmatrix}4 & -4\\ -1 & 5\end{pmatrix}$

$$\begin{pmatrix}4 & -4\\ -1 & 5\end{pmatrix}^{-1} = \frac{1}{16}\begin{pmatrix}5 & 4\\ 1 & 4\end{pmatrix} = \begin{pmatrix}\frac{5}{16} & \frac{4}{16}\\ \frac{1}{16} & \frac{4}{16}\end{pmatrix} = \begin{pmatrix}\frac{5}{16} & \frac{1}{4}\\ \frac{1}{16} & \frac{1}{4}\end{pmatrix}$$

Example 2. State whether $M = \begin{pmatrix}-2 & 4\\ 1 & -1\end{pmatrix}$ has an inverse. If the inverse exists, find it.

Solution

$\det M = (-2)(-1) - (1)(4) = 2 - 4 = -2 \neq 0$, so M^{-1} exists.

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix}-1 & -4\\ -1 & -2\end{pmatrix} = \frac{1}{-2}\begin{pmatrix}-1 & -4\\ -1 & -2\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & 2\\ \frac{1}{2} & 1\end{pmatrix}$$

Example 3. Solve the matrix equation $\begin{pmatrix}3 & 1\\ 3 & 2\end{pmatrix} X = \begin{pmatrix}0 & 7\\ 9 & 2\end{pmatrix}$ for 2×2 matrix X.

Solution.

Let $\begin{pmatrix}3 & 1\\ 3 & 2\end{pmatrix} = A$ and $\begin{pmatrix}0 & 7\\ 9 & 2\end{pmatrix} = B$

The given matrix equation is $AX = B$. If A^{-1} exists we pre-multiply each side of the equation by A^{-1}, and get

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$

So, $X = A^{-1}B$

To find A^{-1}, we first find det A.

$\det A = 3(2) - 3(1) = 6 - 3 = 3 \neq 0$, so that A^{-1} exists

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}$$

Therefore

$$X = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 0 & 7 \\ 9 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -9 & 12 \\ 27 & -15 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 9 & -5 \end{pmatrix}$$

Example 4. $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. Write down inverse matrices A^{-1} and B^{-1}. Hence use your result to find the matrices P and Q such that (i) AP = B, (ii) QA = B.

Solution

$\det A = 4 - 3 = 1$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$\det B = 2 - 12 = -10$

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}$$

(i)

$$\begin{aligned} AP &= B \\ A^{-1}AP &= A^{-1}B \\ IP &= A^{-1}B \\ P &= A^{-1}B = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 2 \\ 5 & 0 \end{pmatrix} \end{aligned}$$

(ii)

$$\begin{aligned} QA &= B \\ QAA^{-1} &= BA^{-1} \\ QI &= BA^{-1} \\ Q &= BA^{-1} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 4 & -5 \end{pmatrix} \end{aligned}$$

Exercise 6.9

1. Find the inverses of the following matrices where possible.

(1) $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$ (3) $\begin{pmatrix} 2 & -1 \\ 3 & -\frac{3}{2} \end{pmatrix}$

(4) $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix}$ (5) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (6) $\begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$

2. Solve each of the following matrix equations for 2×2 matrix X.

(a) $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & 2 \\ 1 & 5 \end{pmatrix} X = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}$

$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{2} & k \\ 0 & 2a \end{pmatrix}$ and $C = \begin{pmatrix} 6 & 2 \\ -3 & h \end{pmatrix}$

(i) If $AB = I$, find the value of k and a.

(ii) Find the value of h for which $\det A = \det C$.

(ii) If the $\det B = \det C$, find the value of h when $a = 3$.

4. Given that $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$, write down the inverse matrix of A. Use your result to find the matrices P and Q such that (i) $AP = B$, (ii) $QA = B$.

5. Given that $A = \begin{pmatrix} 7 & 5 \\ 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, write down the inverse matrix B and use it to find the matrices P and Q such that

(i) $PB = A$, (ii) $BQ = 2A$.

6. Given that $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ -2 & 1 \end{pmatrix}$, write down the matrix A^{-1} and use it to solve the following equations :

(i) $AX = B - A$ (ii) $YA = 3B + 2A$.

6.9 Using Matrices to Solve System of Linear Equations

Consider the following system of equations in which x and y variable on the set of real numbers.

$$\left.\begin{aligned} 3x + y &= 9 \\ 3x + 2y &= 12 \end{aligned}\right\} \text{..................(1)}$$

Since $\begin{pmatrix} 3x + y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, system (1) may be written as a single matrix equation :

$$\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} \text{..................(2)}$$

If we can find an equation equivalent to (2) of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

the solution of the system can be written down at once. To do this, we make use of the fact that the product of a matrix and its inverse is the identity matrix I and proceed as follows :

For matrix $\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$, determinant $= 3 \times 2 - 1 \times 3 = 6 - 3 = 3$,

so $$\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}$$

Pre-multiplying both sides of (2) by the inverse $\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}$,

$$\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 9 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 18-12 \\ -27+36 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Hence $x = 2$ and $y = 3$, which gives $\{(2,3)\}$ as the solution set of the system.

Replacing x by 2 and y by 3 in (1) readily verifies that $\{(2, 3)\}$ is the solution set of the system. This check is always worth making.

Example 1. Find the solution set of the system of equations

$$\left.\begin{aligned} 3x - 7y &= 35 \\ x + y &= 5 \end{aligned}\right\}$$ by matrix method; the variables are on the set of real numt

Solution

$$\left.\begin{aligned} 3x - 7y &= 35 \\ x + y &= 5 \end{aligned}\right\} \quad \text{.....................(1)}$$

System (1) may be written as a single matrix equation :

$$\begin{pmatrix} 3 & -7 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 35 \\ 5 \end{pmatrix} \quad \text{..................(2)}$$

Let $A = \begin{pmatrix} 3 & -7 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 35 \\ 5 \end{pmatrix}$

Then (2) becomes

$$AX = B \quad \text{........................(3)}$$

$$A^{-1} AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\det A = 3 - (-7) = 10$$

Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix}$$

$$X = \frac{1}{10} \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 35 \\ 5 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 35+35 \\ -35+15 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 70 \\ -20 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Therefore $x = 7$, $y = -2$

The solution set of the system is $\{ (7, -2) \}$

Example 2. Try to solve the system of equations $x + y = 4$

$3x + 3y = 12$

Explain, with the aid of a cartesian diagrams, why you failed.

Solution

The given system of equations

$$\left.\begin{aligned} x + y &= 4 \\ 3x + 3y &= 12 \end{aligned}\right\} \text{..................(1)}$$

System (1) may be written as a single matrix equation

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} \quad \text{......................(2)}$$

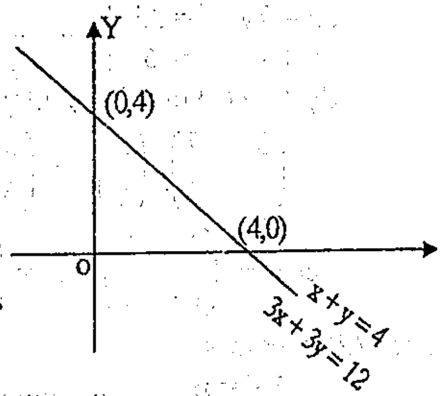
det of $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = 1\,(3) - 3\,(1) = 0$.

So inverse of $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$ does not exist.

$\therefore$ equation (2) cannot be solved by matrix method.

Geometrically speaking, both equations of system (1) represent the same straight line, as shown in the figure.

Therefore the point of intersection of the two straight lines is not unique, and hence we cannot have a unique solution for x and y.



Exercise 6.10

Find the solution set of the systems of equations in question 1 – 12 by matrix method .The variables are on the set of real numbers.

1. $x - y = 7$, $x + y = 11$
2. $x + 3y = 6$, $2x + y = 4$
3. $5x + 6y = 25$, $3x + 4y = 17$
4. $2x - 5y = 1$, $3x - 7y = 2$
5. $6x + 7y = 4$, $5x + 6y = 3$
6. $3x + 2y = 7$, $5x - y = 3$
7. $4x + 5y = 0$, $2x + 5y = 1$
8. $x + y = 3$, $y - x = 1$
9. Find the inverse of the matrix $\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}$ and use it to solve the following systems. $7x + 8y = 10$, $5x + 6y = 7$.
10. Find the inverse of the matrix $\begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix}$ and use it to solve the simultaneous equations, $3x + 4y = 18$ and $2x + 6y = 22$.

11. Find the inverse of the matrix $\begin{pmatrix} 7 & 4 \\ 3 & 2 \end{pmatrix}$. Hence determine the coordinates of the point of intersection of the lines $7x + 4y = 16$ and $3x + 2y = 6$.

12. Try to solve $9x + 6y = 4$ and $6x + 4y = 2$ by matrices. Explain with the aid of a Cartesian diagram, why you failed.

SUMMARY

(1) A **matrix** is a rectangular array of numbers arranged in rows and columns, the array being enclosed in round (or square) brackets. The numbers are called entries or elements.

(2) The **order** of a matrix is given by the number of rows followed by the number of columns.

e.g $\begin{pmatrix} 3 & 1 & 7 \\ 4 & \text{[illegible]} & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 7 \\ 9 & 4 \end{pmatrix}$ are respectively order 2×3, order 2×2 or a square matrix of order 2.

(3) Two matrices are **equal** if and only if they are of the same order and their corresponding entries are equal.

(4) A **zero matrix** O, is a matrix whose elements are all zero.

(5) A **unit matrix** I is a square matrix whose elements in the main diagonal are unity and whose other elements are all zero.

e. g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(6) **Addition of matrices**

If A and B are two matrices of the same order, the sum of A and B, denoted by A + B, is the matrix obtained by adding each entry of A to the corresponding entry of B.

$$\begin{array}{cccc} & A & B & A+B \\ \text{e. g.} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} + & \begin{pmatrix} p & q \\ r & s \end{pmatrix} = & \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix} \end{array}$$

(7) The **negative** of the matrix A, written $-A$, is the matrix whose entries are the negatives of the entries in A.

e. g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow -A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

(8) **Multiplication of matrices by real numbers (scalars)**

To multiply a matrix by a real number k, we multiply each entry by that number.

e.g. $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

This operation is scalar multiplication.

(9) **Multilication of two matrices.**

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$

Rule : Multiply "row into column and add the products".

Multiplication of matrices is not in general commutative; it is associative, and distributive with respect to matrix addition.

(10) **Inverse of a 2×2 matrix**

The inverse of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix

$A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided that $ad - bc \neq 0$, $ad - bc$ is the determinant of matrix A.

If det A = 0, A has no inverse, and is said to be a singular matrix.

If det A $\neq$ 0, then A is said to be non-singular.

Property : $A^{-1}A = AA^{-1} = I$

(11) **Matrix equations**

If A, B and X are square matrices of the same order, such that $AX = B$ and A has an inverse A^{-1}, then

$$AX = B \Leftrightarrow A^{-1}AX = A^{-1}B \Leftrightarrow IX = A^{-1}B \Leftrightarrow X = A^{-1}B$$

CHAPTER 7
Introduction to Probability

The probability of an event in a random experiment was introduced in Grade 10. In this chapter, we will consider more about probability. First we recall that the probability of an event is a number between 0 and 1 defined by

$$\text{probability of an event} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

The closer the probability of an event is to 1, the more likely it is that the event will occur. The closer the probability is to 0, the less likely it is that the event will occur. If an event is certain not to occur, then its probability is 0.

In the following, we will describe how to calculate the probability by drawing tree diagrams, or by constructing tables. We will also explain how the expected frequency of an event in a given number of repetitions of an experiment can be measured.

7.1 Calculating Probabilities by Using Tree Diagrams

In this section, we will use tree diagrams to find the probability of outcomes in an experiment. In many cases the outcomes of an experiment can be ordered pair or ordered triples of numbers or objects. In such cases, to tabulate all possible outcomes it is sometimes helpful to use a tree diagram. For example, suppose one coin is tossed two times and the result of each toss (heads or tails) is recorded. To find the possible outcomes of the pair toss, the following tree diagram is used.

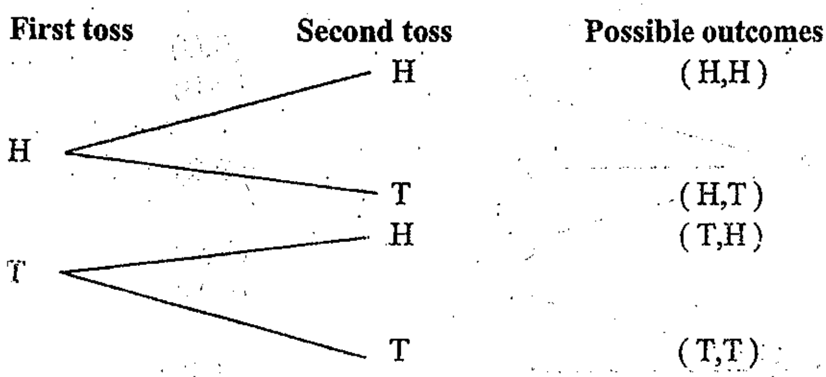


Fig. 7.1

Fig 7.1 shows four open branches . Each open branch represents an outcome. Thus, the set of all possible outcomes is given by

$\{ (H, H), (H,T), (T, H), (T, T) \}$

If we want to find the probability of obtaining two heads, we see that one open branch is a favourable outcome as given in Fig. 7.2.

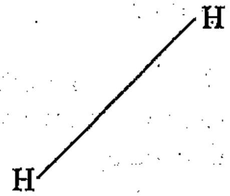


Fig. 7.2

This gives the required probability as

$$P(\text{two heads}) = \frac{\text{no. of favourable outcomes}}{\text{no. of all possible outcomes}} = \frac{1}{4}$$

Example 1. Suppose a box contains 3 marbles 1 black, 1 red and 1 green. A marble is chosen and the colour is recorded. This marble is replaced before a second marble is chosen. Find the probability of choosing two different colours.

Solution

The following tree diagram can be used to determine the possible outcomes.

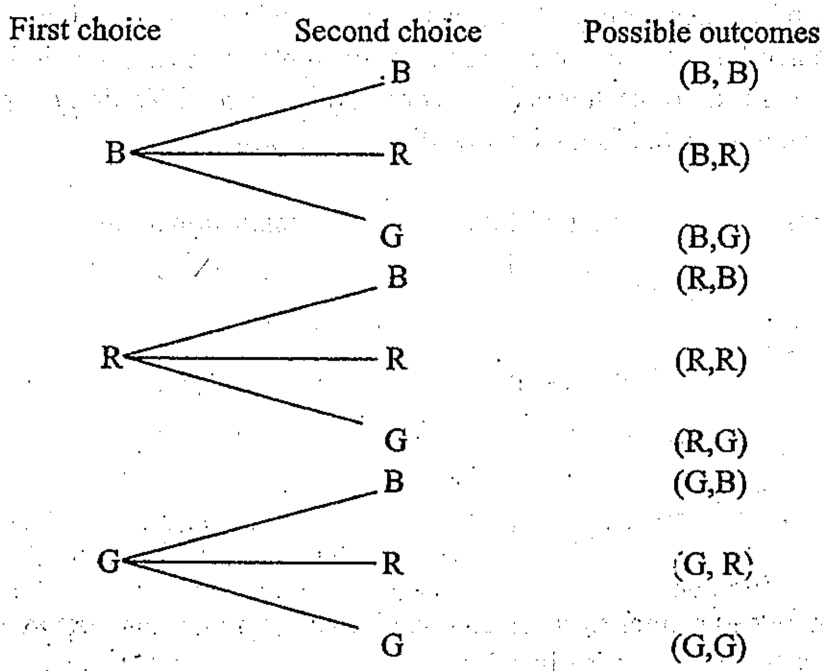


Fig. 7.3

The set of all possible outcomes is given by

{ (B,B), (B,R),(B,G),(R,B), (R,R),(R,G),(G,B),(G,R),(G,G)}

This set contains nine outcomes.

If we denote the set of favourable outcomes by A, we have

A = { (B,R),(B,G),(R,B), (R,G),(G,B),(G,R)}.

Since this set contains six outcomes, the probability of event A is given as

$$P(A) = \frac{6}{9} \text{ or } \frac{2}{3}$$

Example 2. How many 3 digit numerals can you form from 1, 5 and 7, without repeating any digit ? Find the probability of a numeral which begins with 1.

Solution

We use the following tree diagram to determine all possible outcomes.

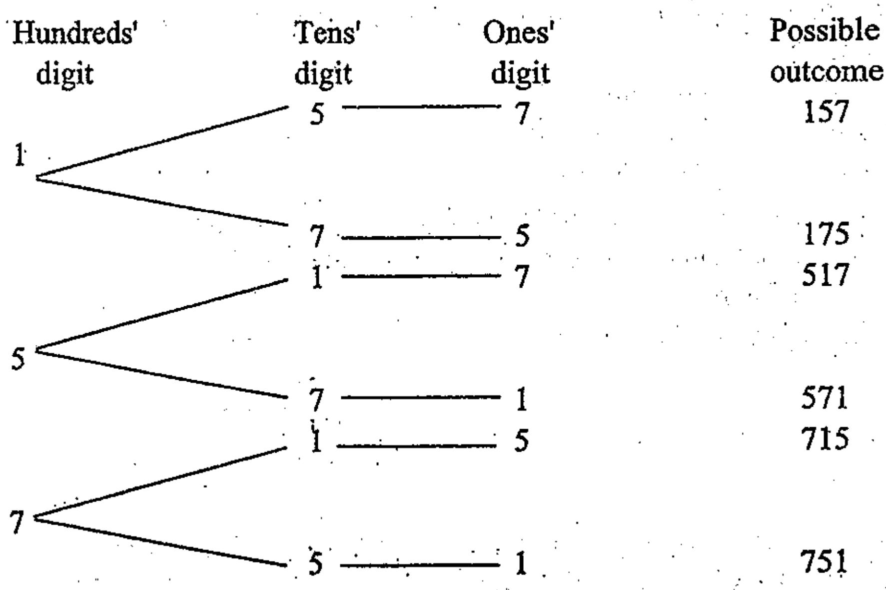
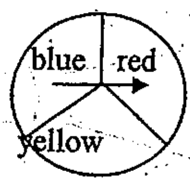


Fig. 7.4

The tree diagram shows that we can form six numerals from the given digits. There are only two open branches for the favourable outcomes. Thus, the probability of a numeral which begins with 1 is given as $\frac{2}{6}$ or $\frac{1}{3}$.

Exercise 7.1

1. Maung Ba, Maung Hla and Maung Mya are candidates for president of a badminton team. Ma Ni, Ma Yi, Ma Thi and Ma Si are candidates for vice-president. Draw a tree diagram to determine the set of all possible outcomes. Find the probability that Maung Ba is to be elected for president.
2. A box contains 5 cards numbered as 2, 3, 4, 5 and 9. A card is chosen, the number is recorded, and the card is replaced. Then another card is chosen and the number is recorded. Draw a tree diagram and tabulate possible outcomes. Find the probabilities of
   (a) getting two prime numbers
   (b) getting two odd numbers and
   (c) getting a pair of numbers where the sum is a prime number.
3. Suppose the first card is not replaced in the problem 2. Another exercise (a) through (c).
4. A box contains 4 marbles of 2 blue, 1 red and 1 yellow. A marble is chosen, the colour is recorded, and the marble is not replaced. Then another marble is chosen and the colour is recorded. Draw a tree diagram to determine possible outcomes.
   Hence, find the probabilities of
   (a) choosing 2 blue marbles and
   (b) choosing 2 different colours.
5. Spin the arrow twice and record the colour you get each time.
   Draw a tree diagram to list possible outcomes. Hence, find the probability of
   (a) not spinning red first,
   (b) spinning two different colours.



6. Maung Maung , Mg Mya, Ma Hla and Ma Khin are finalists in a mathematics contest . One of these pupils will win first prize, and another will win second prize. In how many ways is it possible for the first and second prize winners to be chosen from the four pupils? Find the probability that Maung Mya and Ma Khin both win prizes.
7. A coin is tossed three times. Head or tail is recorded each time. Drawing a tree diagram, find the probability of

(a) getting exactly one head and

(b) getting no heads.

8. Suppose a family has 3 children. Find the probability that the first two children born are boys. What is the probability that the last two children are boys?

9. A coin is tossed and then a die is thrown. Head or tail and the number turns up are recorded each time. Draw a tree diagram and list the possible outcomes. Hence, find the probability that head and 6 turn up.

7.2 Combinations of Outcomes

In this section, we will present some combinations of outcomes of an experiment. We will do this by given simple examples. Let us first consider the following experiment.

Example 1. Suppose that a blue die and a black die are rolled.
The set of all possible outcomes is shown in the table below.

black die

		1	2	3	4	5	6
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
blue die	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

From the table, a total score of 5 can be obtained in 4 ways namely

(4,1), (3,2),(2,3),(1,4). Hence the probability of getting a total of 5 is $\frac{4}{36}$, i.e,

$$P(5) = \frac{4}{36} = \frac{1}{9}$$

Also from the table, the probability of not getting 5 is $\frac{32}{36}$, i.e,

$$P(\text{not } 5) = \frac{32}{36} = \frac{8}{9}$$

Notice that

$$P(5) + P(\text{not } 5) = \frac{1}{9} + \frac{8}{9} = 1$$

This form of result is generally true for any outcome of experiment. This result can be stated as follows:

If the probability of an outcome of an experiment is P, then the probability that the outcome will not happen is 1 – P.

Example 2. Again, let us consider the outcomes of the experiment in example 1.

From the table of example 1, we have,

$$P(10) = \frac{3}{36} = \frac{1}{12} \text{ and } P(5 \text{ or } 10) = \frac{7}{36}.$$

Notice that

$$P(5 \text{ or } 10) = P(5) + P(10)$$

This form of result can also be used in general when the sets of outcomes are quite separate, that is, when there is no member which appears in both sets.

In this case we say that the outcomes are **mutually exclusive.**

If A and B are mutually exclusive outcomes, then

$$\mathbf{P(A \text{ or } B) = P(A) + P(B).}$$

Example 3. Again a blue die and a black die are rolled, and the possible outcomes are shown in the table below.

				black die			
		1	2	3	4	5	6
	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
blue die	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

From the table, P (blue 4) = $\frac{6}{36} = \frac{1}{6}$, the set of favourable outcomes is shown in the horizontal box. Also P (black 4) = $\frac{6}{36} = \frac{1}{6}$; the set of favourable outcomes is shown in the vertical box. Also P (blue 4 and black 4) = $\frac{1}{36}$, the set of favourable outcomes is shown in the intersection of the two boxes.

Notice that

P (blue 4 and black 4) = P (blue 4) × P (black 4)

This result is generally true when the two outcomes occur independently of each other (in this case the blue 4 and the black 4 occur quite independently of each other).

If A and B are outcomes which are independent of each other,

P (A and B) = P (A) × P (B).

Example 4. Three tennis players A, B, C play each other only once. The probability that A will beat B is $\frac{1}{3}$, that B will beat C is $\frac{2}{5}$ and that C will beat A is $\frac{2}{7}$. Calculate the probability that C wins both games.

Solution. P (C wins both games) = P (C beats A and C beats B)

= P (C beats A) × P (C beats B)

= P (C beats A)×(1 – P **(B beats C)**)

$= \frac{2}{7} \times (1 - \frac{2}{5}) = \frac{6}{35}$.

Example 5 Three groups of people are comprised as follows

Frist group	3 women	2 men
Second group	3 women	3 men
Third group	3 women	24 men

One person is selected at random from each group. Calculate the probability that the three people selected are all women.

Solution:

P (the three people selected are all women)

= P (the person from the first group is a woman **and** the person from the second group is a

woman **and** the person from the third group is a woman)

= P (the person from the first group is a woman) × P (the person from the second group is a

woman)× P (the person from the third group is a woman)

$= \frac{3}{5} \times \frac{3}{6} \times \frac{3}{7} = \frac{9}{70}$

Exercise 7.2

1. (a) Write down the set of all possible outcomes for the rolling of two dice and find the probabilities for the total scores on the two dice.

   P (2), P (3), P (4), ... , P (12).

   (b) Are all of these outcomes equally likely?
2. What is the most likely, and the least likely, score on rolling two dice?
3. What is P (the total score is 2 or 12) for rolling two dice?
4. What is P (the total score is 3 or 4 or 5) for rolling the two dice?
5. What is P (the total score is prime number) for rolling the two dice?
6. What is P (the total score is greater than 7) for rolling the two dice?
7. A blue die and a black die are rolled. Find the probability of getting a score which

   (a) includes a 1 on the blue die,

   (b) includes a 1 on the blue die or a 6 on the blue die,

   (c) includes a 2 on the blue die or a 5 on the black die.

8. When two dice are rolled, what is the probability of an outcome in which the score on the second die is greater than that on the first?

9. Construct the table of outcomes for rolling two dice, a blue die and a black die, and use it to find P (blue 2 and black 5). Find also P (blue 2 and black 5) by using a property of independent outcomes.

10. Construct the table of outcomes for rolling two dice, a blue die and a black die, and use it to find P(blue 1 and black number greater than 4). Find also P(blue 1 and black number greater than 4) by using a property of independent outcomes.

11. Copy and complete the table for the toss of a coin and the roll of a die.

		Die					
		1	2	3	4	5	6
Coin	H	(H,1)	(H,2)	.	.	.	.
	T	.	.	.	.	.	(T,6)

(a) How many members are there in the set of possible outcomes?
(b) Show by a box the subset of outcomes containing a Tail.
(c) Show by a box the subset of outcomes containing 4.
(d) From the table, what is P(Tail), and what is P (4)?
(e) Verify that P (Tail,4) = P (Tail) × P (4).

12. Make a table for the toss of two coins, putting first coin on the left, second coin at the top. Find P (H,H), P (T,T) and P (a head and a tail in any order).

13. A spinner is equally likely to point to any one of 1, 2, 3, 4 . Make a table of ordered pairs (First spin, Second spin). Find the probability of:
(a) two even numbers,
(b) two odd numbers,
(c) an even number followed by an odd number,
(d) an odd number followed by an even number.

14. The spinner as in question 13 is spun once then a die is rolled.
Make a table of ordered pair (Spinner, Die). Hence , find (where E means even and O means odd).
(a) P (E,E)
(b) P ((E,O) or (O,E))
(c) P (total of 10)
(d) P (total of 1)
(e) P (total less than 6)

15. Copy and complete this array of ordered triples for the possible outcomes when 3 coins are tossed simultaneously:

HHH	HHT	HTH	HTT
THH			

Hence, find the probability of getting :
(a) exactly 2 Heads,
(b) 2 Heads and a Tail in any order,
(c) 3 Tails.

16. A bag contains 15 discs of which 3 are white, 5 are red and 7 are blue. Two discs are to be drawn at random, in succession, each being replaced after its colour has been noted. Calculate the probability that the two discs will be of the same colour.

17. The probabilities that the students A and B will pass an examination are $\frac{2}{3}$ and $\frac{3}{4}$, respectively. Find the probabilities that
(i) both A and B pass the examination,
(ii) exactly one of A and B passes the examination.

18. Three groups of children consist of 3 boys and 1 girl, 2 boys and 2 girls, and 1 boy and 3 girls respectively. If a child is chosen from each group, find the probability that 1 boy and 2 girls are chosen.

7.3 Calculation of Expected Frequency

In this section, we will give a method of calculating expected frequency. Expected frequency of an outcome in an experiment is useful when we need to repeat such an experiment many times. If we repeat an experiment many times, then the number of times we expect a favourable outcome to turn up is a helpful information for us. For example, if we roll a die 300 times, the times we expect 3 to turn up can be calculated as follows:

In this case $P(3) = \frac{1}{6}$, thus,

number of 3, expected in 300 throws $= \frac{1}{6} \times 300 = 50$

Of course, we would not be surprised if, in such an experiment, the number of times 3 turned up was 47,or 52. But we would be very surprised if the number of times was 2 or 290.

As in the example, we can calculate the expected frequency by using the following definition ; that is,

In a number of trials,

the expected frequency of an outcome

= (the probability of the outcome) × (the number of trials).

Example 1. If a coin is tossed 100 times, what is the expected frequency of ' Head'.

$P(\text{Head}) = \frac{1}{2}$

Number of Head expected in 100 tosses $= \frac{1}{2} \times 100 = 50$

Example 2. It was found that the probability of a child getting measles was 0.13. Out of 1200 children , how many would you expect to catch measles?

P (a child getting measles) = 0.13

Number of children expected getting

measles in 1200 children $= 0.13 \times 1200 = 156$

Exercise 7.3

1. After a large number of trials tossing drawing pins the probability of ' Pin up ' was estimated to be 0.3. In 400 more trials, how many times would ' Pin up ' be expected?

2. If a die is rolled 60 times, what is the expected frequency of

   (a) 1 turns up.

   (b) a number divisible by 3 turns up.

   (c) a factor of 6 turns up.

3. (a) List the set of four possible outcomes when two coins are tossed.

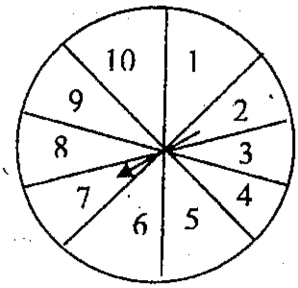
(b) How many would you expect to obtain two heads in 200 trials?

(c) Would you be surprised to obtain two heads

(i) 53 times (ii) 185 times (iii) not at all.

4. If the arrow in the given figure is spun 100 times, what is the expected frequency of:

(a) a 10 (b) an odd number.



5. If the arrow is spun 1000 times, what final score would you expect if all the individual scores are added together?

6. The probability of scoring 12 when throwing two dice at once is $\frac{1}{36}$. If such experiment is repeated 720 times, what would you expect if the score not being 12?

7. A spinner is equally likely to point to any one of the number 1, 2, 3, 4, 5, 6, 7. What is the probability of scoring a number divisible by 3? If the arrow is spun 700 times, how many would you expect scoring a number not divisible by 3?

SUMMARY

A tree diagram or a table of possible outcomes are very useful in finding the probabilities.

We say that the outcomes are mutually exclusive if they cannot occur together. If A and B are mutually exclusive outcomes, then P (A or B) = P (A) + P (B).

Two outcomes are said to be independent if the occurrence of one outcome does not affect the probability of the other. If A and B are indepnedent outcomes, then P (A and B) = P (A) $\times$ P (B).

In a number of trials,

the expected frequency of an outcome

= (the probability of the outcome) $\times$ (the number of trials)

CHAPTER 8

Circles

In the Grade 10 Text, we learned about the chords, secants and tangents related to the circle. Now, we will learn the relationships between circles and angles.

8.1 Angles in a Circle

Theorem i

The angle which an arc of a circle subtends at the centre is double of that which it subtends at any point on the remaining part of the circumference.

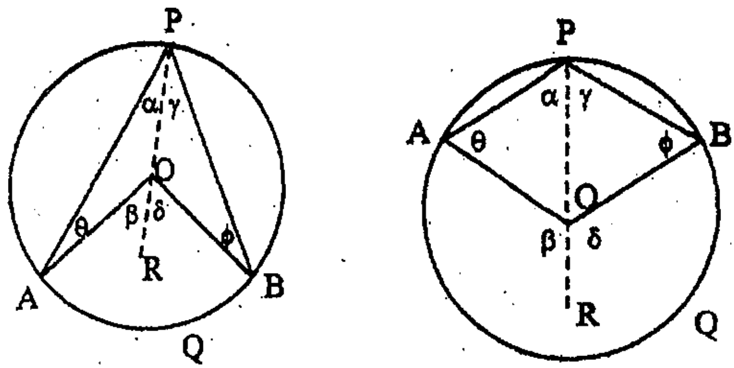


Fig. 8.1

Given: $\odot$ APB (centre O), in which arc AQB subtends $\angle$AOB at the centre and $\angle$APB at point P on the remaining part of the circumference.

Prove: $\angle AOB = 2\angle APB$.

Proof: Join PO and produce it to R.

In ΔOPA, OP = OA (radii)

$$\alpha = \theta$$

But β is an external angle ;

$$\therefore \beta = \alpha + \theta$$

$$\therefore \beta = 2\alpha$$

In the same way, $\delta = 2\gamma$;

$$\therefore \quad \beta + \delta = 2\alpha + 2\gamma = 2(\alpha + \gamma)$$

i.e, $\angle AOB = 2 \angle APB$.

Corollary 1.1 Angles in the same segment of a circle are equal to one another.

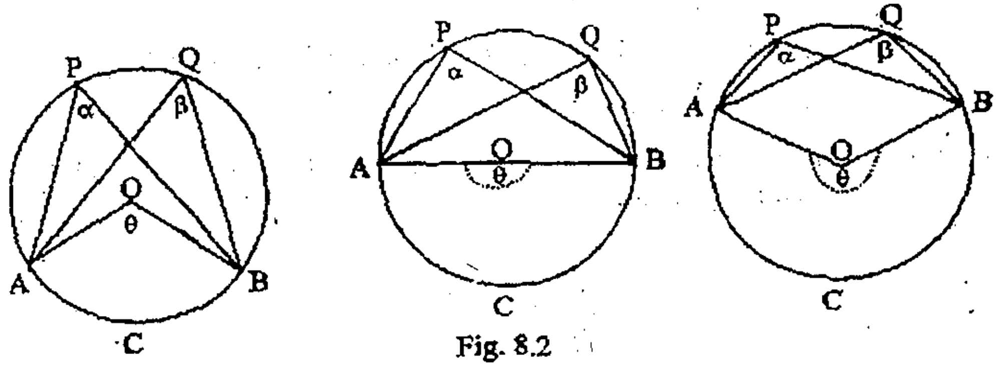


Fig. 8.2

Given : $\odot APB$ (centre O), in which α and β are angles in the same segment APQB.

Prove : $\alpha = \beta$

Proof : Join OA, OB (forming angle θ at the centre)

Since α, β and θ stand on the same arc ACB,

$$\alpha = \frac{1}{2}\theta \text{ and } \beta = \frac{1}{2}\theta$$

$$\therefore \alpha = \beta$$

Corollary 1.2 The angle in a semicircle is a right angle.

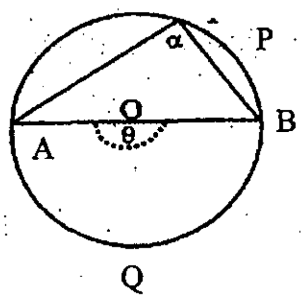


Fig. 8.3

Given : Angle α in the semicircle APB

Prove : α = one right angle

Proof: Complete ⊙APBQ

θ (at the centre) and α (at the circumference) stand on the same arc AQB.

$$\therefore\ \theta = 2\alpha$$

But θ is a straight angle = 2 rt. ∠s ; (2 right angles)

$\therefore\ \alpha = 1$ rt. angle

Corollary 1.3 The opposite angles of a quadrilateral inscribed in a circle are supplementary.

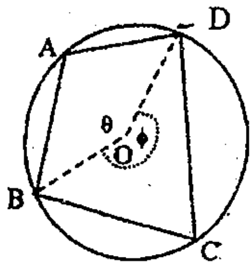


Fig. 8.4

Given: Quadrilateral ABCD inscribed in circle O.

Prove : $\angle A + \angle C = 2$ rt. ∠s and $\angle B + \angle D = 2$rt. ∠ s

proof : Join OB, OD

$\angle C = \frac{1}{2}\theta$ (standing on arc BAD)

$\angle A = \frac{1}{2}\phi$ (standing on arc BCD)

$$\therefore\ \angle C + \angle A = \frac{1}{2}(\theta + \phi)$$

But $\theta + \phi = 4$ rt. ∠s;

$\therefore\ \angle C + \angle A = 2$ rt. ∠s

Similarly, $\angle B + \angle D = 2$rt. $\angle$s.

Corollary 1.4 If one side of a quadrilateral inscribed in a circle is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

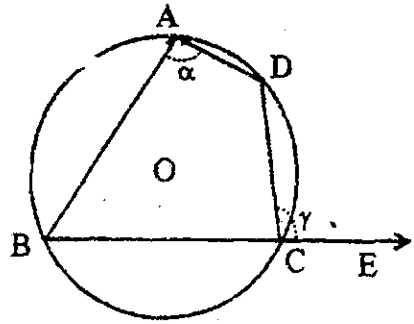


Fig. 8.5

Given: Quadrilateral ABCD inscribed in $\odot$ O and BC is produced to E.

Prove; $\gamma = \alpha$

Proof: Since ABCD is inscribed in $\odot$ O, $\alpha + \angle BCD = 180°$

But $\angle BCD + \gamma = 180°$

$\therefore \alpha + \angle BCD = \angle BCD + \gamma$

Hence $\alpha = \gamma$

Theorem 2

In congruent circles, or in the same circle, equal angles at the centre stand on equal arcs.

Conversely, in congruent circles or in the same circle, equal arcs subtend equal angles at the centre.

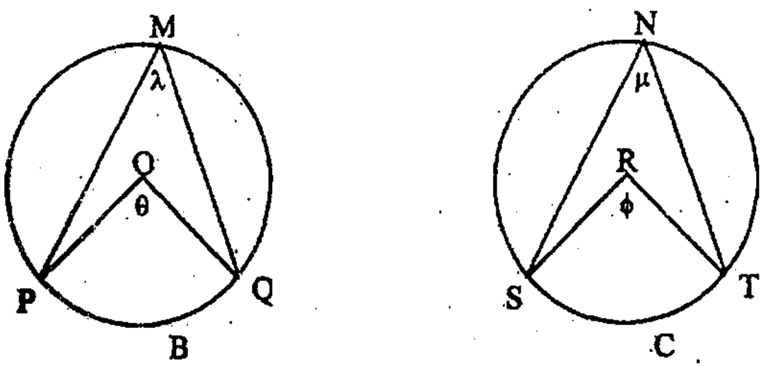


Fig. 8.6

ven: Congruent circle PMQ, SNT (centres O and R respectively) with

$\theta = \phi$ (angles at the centres)

ve: arc PBQ = arc SCT.

of: Apply $\odot$PMQ to $\odot$SNT so that centres O are R coincides ,

and OP falls along RS.

Then OP = RS (radii of congruent circles)

and P will fall on S.

Since $\theta = \phi$, OQ will fall along RT and since OQ = RT, Q will fall on T.

Since the circles are congruent and their centres O and R coincide, their circumferences coincide, and since P falls on S and Q on T, arc PBQ coincides with arc SCT;

$\therefore$ arc PBQ = arc SCT.

Converse

en: Congruent circles PMQ, SNT (centres O and R respectively), with arc PBQ = arc SCT, and let θ and ϕ be the angles subtended by their arcs at the centres.

ve: $\theta = \phi$

of: As before, apply $\odot$PMQ to $\odot$SNT so that there centres coincide.

Then, since the circles are congruent their circumferences will coincide, and if OP falls along RS, P will fall on S ($\because$ OP = RS).

And since arc PBQ = arc SCT, Q will fall on T.

$\therefore$ OQ will fall on RT

$\therefore$ $\theta = \phi$

rollary 2.1

In congruent circles, or in the same circle, equal angles at the circumference id on equal arcs , and conversely , equal arcs subtend equal angles at the umference.

$$[\lambda = \frac{1}{2}\theta \text{ and } \mu = \frac{1}{2}\theta \,;\, \lambda = \mu \Leftrightarrow \theta = \phi \Leftrightarrow \text{arc PBQ} = \text{arc SCT}]$$

Theorem 3

In congruent circles , or in the same circle , equal chords cut off equal arcs. Conversely, in congruent circles, or in the same circle, the chords of equa arcs are equal.

Fig. 8.7

Given : Congruent circles PMQ, SNT (centres O, R respectively) with PQ = ST.

Prove : arc PBQ = arc SCT, and arc PMQ = arc SNT.

Proof: Join OP, OQ and RS , RT.

In Δ s OPQ , RST,

OP = RS (equal radii)

OQ = RT (equal radii)

PQ = ST (given)

Δ OPQ $\cong$ Δ RST (SSS congruency)

$\therefore$ $\beta = \delta$

$\therefore$ arc PBQ = arc SCT (Th.2)

and since the circles are congruent, arc PMQ = arc SNT.

Converse

Given: Congruent circles PMQ, SNT (centres O, R respectively) with arc PBQ = arc SCT.

Prove : PQ = ST

Proof: Join OP, OQ and RS, RT.

Since the circles are congruent and arc PBQ = arc SCT (Th. 2)

$\beta = \delta$

In Δs OPQ, RST,

$OP = RS$ (equal radii)

$OQ = RT$ (equal radii)

$\beta = \delta$ (proved)

$\therefore \Delta OPQ \equiv \Delta RST$ (SAS congruency)

$\therefore PQ = ST$

N.B. In this and the preceding theorem it is left as an exercise for the student to adapt the proofs to meet the case of the same circle.

Theorem 4

The angles which a tangent to a circle makes with a chord drawn through the point of contact are equal to the angles in the alternate segments of the circle.

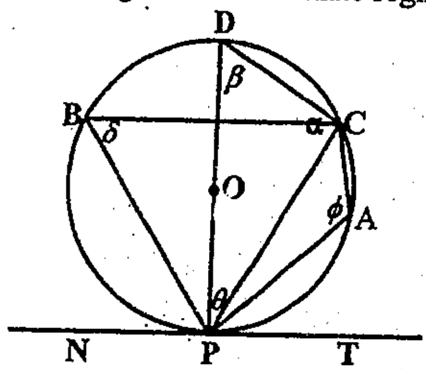


Fig. 8.8

Given: $\odot$DPC, with NT the tangent at P and PC a chord through P.

Prove: (i) $\angle CPT = \delta$. (ii) $\angle CPN = \phi$.

Proof: From point P draw diameter PD. Join BC. In minor arc CP take any point A. Join DC, CA, AP.

(i) $\alpha = 90°$ (PD is a diameter)

$\therefore \theta + \beta = 90°$

Also $\theta + \angle CPT = \angle DPT = 90°$

$\therefore \theta + \angle CPT = \theta + \beta$

$\therefore \quad \angle CPT = \beta$

But $\quad \delta = \beta \quad$ (angle in the same segment)

Hence $\quad \angle CPT = \delta$

(ii) DPAC is a quadrilateral inscribed in ⊙ O

$\therefore \quad \beta + \phi = 180°$

Also $\quad \angle CPT + \angle CPN = 180°$

$\therefore \quad \angle CPT + \angle CPN = \beta + \phi$

But $\quad \angle CPT = \beta \quad$ (proved)

$\therefore \quad \angle CPN = \phi$

Example 1.

Two unequal circles are tangent externally at O. AB is a chord of the first circle. AB is tangent to the second circle at C, and AO meets this circle at E. Prove that $\angle BOC = \angle COE$.

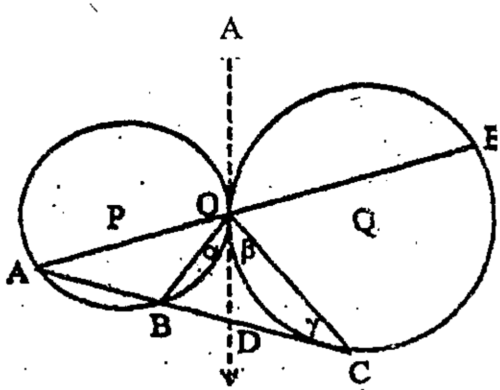


Fig. 8.9

Given: Two unequal circle P and Q are tangent externally at O.AB is a chord of ⊙P, and AB is tangent to ⊙Q at C. AO meets ⊙Q at E.

Prove: $\angle BOC = \angle COE$.

Proof: Draw a common tangent at O cuts BC at D.

$\alpha = \angle A \quad$ (Th.4)

$\beta = \gamma \quad$ (Two equal tangents to ⊙Q from external point D)

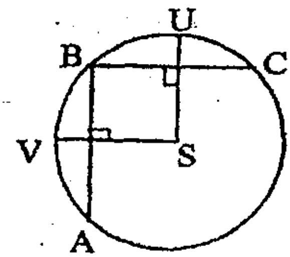
$\therefore \alpha + \beta = \angle A + \gamma$

$= \angle COE$

$\therefore \angle BOC = \angle COE$

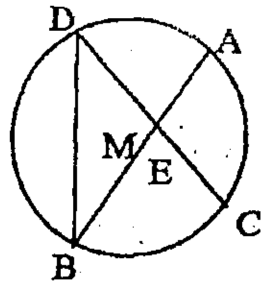
Exercise 8.1

1. In $\odot S$, **AB and BC** are equal chords, SV $\perp$ **AB, and** SU $\perp$ BC. Prove that B is the midpoint of arc VU.



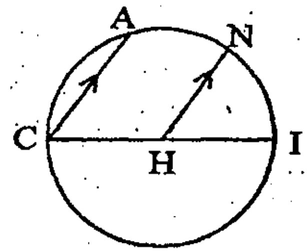
2. Given: $\odot M$, AB = CD

   Prove: Δ DBE is isosceles with base BD.



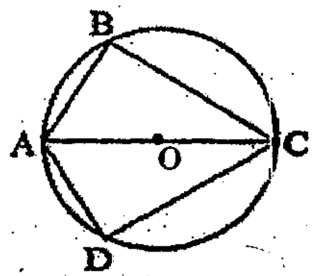
3. $\odot H$ with diameter CI, CA//HN

   Prove that arc AN = arc NI.

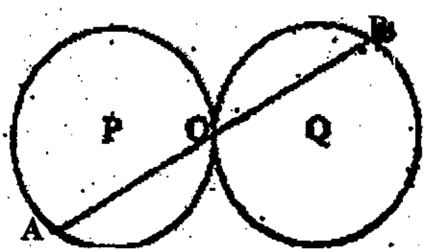


4. Given : $\odot O$ with AB = AD and AC is a diameter.

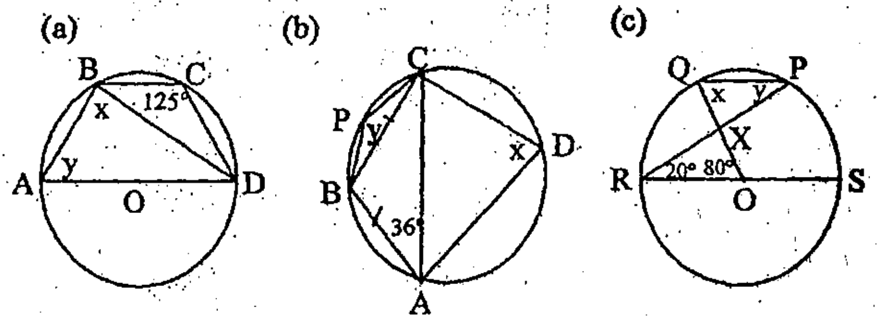
   Prove : BC = CD.



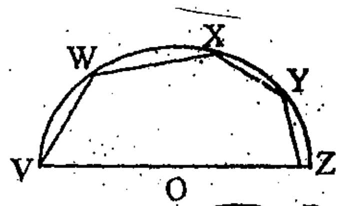
5. Circles P and Q are congruent and tangent externally at O. Prove that OA=OB.



6. In each of the figures, find x and y.



7. VWXYZ is a semicircle, V and Z being the extremities of the diameter and W,X,Y being any points on the arc. Prove that $\angle VWX + \angle XYZ$ = three rt. $\angle$s.



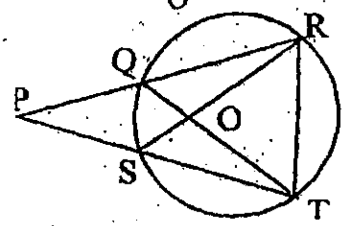
8. In the figure PST and PQR are any two secants drawn from P to the circle.

Prove the following pairs of triangles equiangular: (a) SOT, QOR;

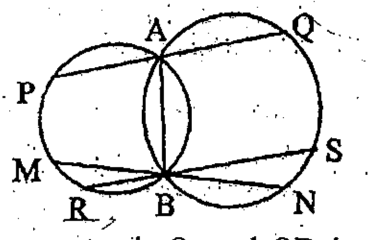
(b) SOQ, TOR;

(c) PSR, PQT;

(d) PSQ, PRT.



9. In the figure PQ is parallel to RS and $\angle PAB = \angle MBA$. Prove PQ = RS = MN.



10. ABC is a triangle inscribed in a circle whose centre is O, and OD is the perpendicular drown O to BC. Prove $\angle BOD = \angle BAC$.

11. Two circles intersect at M, N and from M diameters MA, MB are drawn in each circle. If A, B be joined to N, prove ANB a straight line.

Proof: Join AD , CB.

In Δ APD and Δ CPB

$\alpha = \alpha_1$ (Subtends the same arc BD)

$\beta = \beta_1$ (Subtends the same arc AC)

$\therefore$ $\Delta APD \sim \Delta CPB$ (AA cor :)

$$\frac{AP}{CP} = \frac{PD}{PB}$$

$\therefore$ $AP.PB = CP.PD$

Theorem 6

If a secant and a tangent are drawn to a circle from an external point, the square of the length of the tangent segment is equal to the product of the length of the secant segment and its external part.

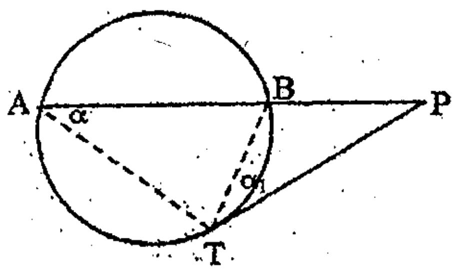


Fig. 8.13

Given: $\odot$ABC with secant PBA cutting it at B and A, and tangent PT touching it at T.

Prove: $PA.PB = PT^2$

P: Join AT, BT

In ΔPAT and Δ PTB

$\alpha = \alpha_1$ (Th.4)

$\angle P$ is common

$\therefore$ $\Delta PAT \sim \Delta PTB$ (AA corollary)

$$\frac{PA}{PT} = \frac{PT}{PB}$$

$\therefore$ $PA.PB = PT^2$

Corollary 6.1

If two chords of a circle intersect at a point without the circle, the product of the lengths of the segments of the one is equal to the product of the lengths of the segments of the other.

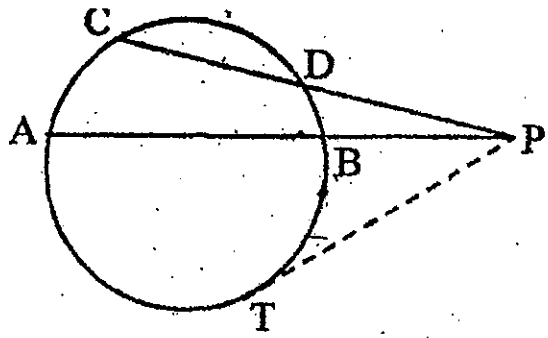


Fig. 8.14

Given: Two chords AB and CD of a circle intersect at a point P without the circle.

Prove: $PA \cdot PB = PC \cdot PD$

Proof: Draw tangent PT.

$$\left.\begin{array}{l} PA \cdot PB = PT^2 \\ PC \cdot PD \end{array}\right\} = PT^2$$

(Th.6)

$\therefore PA \cdot PB = PC \cdot PD$

Example 1.

Two chords of a circle, AB and CD, intersect at right-angles at K. If AK = 6cm, CK= 3cm, and KD = 4cm, find KB. If E is the midpoint of KD and AE is produced to meet the circle again at F, show that AE = 4EF.

Solution

$$AK.KB = CK.KD$$
$$6\,KB = 3 \times 4$$
$$KB = 2 \text{ cm}$$

Since $KE = ED = \frac{1}{2}(4) = 2$ cm,

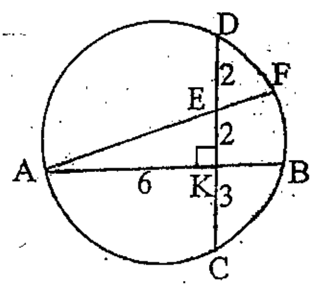


Fig. 8.15

$CE = 3 + 2 = 5$ cm

In rt. ΔAKE, $AE^2 = AK^2+KE^2 = 6^2 + 2^2 = 40$

$$AE = \sqrt{40} = 2\sqrt{10} \text{ cm}$$

$AE.EF = CE.ED$

$2\sqrt{10}\, EF = 5 \times 2$

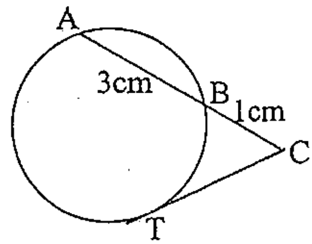
$$EF = \frac{10}{2\sqrt{10}} = \frac{1}{2}\sqrt{10} \text{ cm}$$

$$4EF = 4\left(\frac{1}{2}\sqrt{10}\right) = 2\sqrt{10} \text{ cm}$$

$$AE = 4EF$$

Example 2.

A and B are two points on a circle 3 cm apart. The chord AB is produced to C making BC =1 cm. Find the length of the tangent from C to the circle.



Solution

$CB.CA=CT^2$

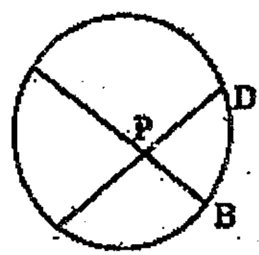
$1(1+3)= CT^2$

$CT^2 = 4$

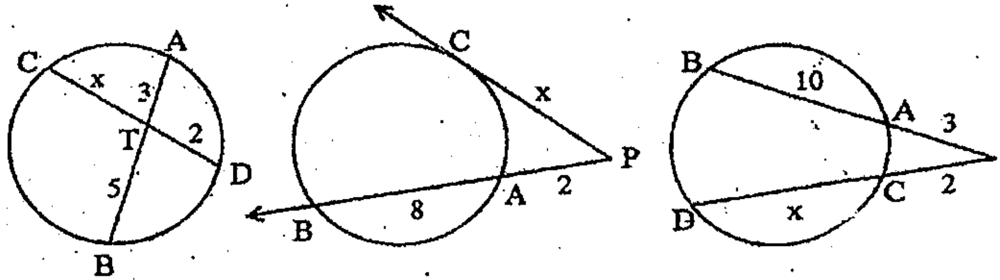
$CT = \sqrt{4} = 2$ cm

Exercise 8.2

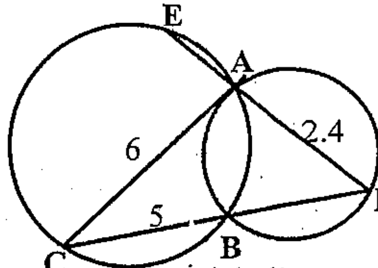
1. In the figure if (a) AP = 10, PC=5, PD =6, find PB; if (b) AP = 10, PD = 6, DA = 12 BC = 9, find AB and CD.



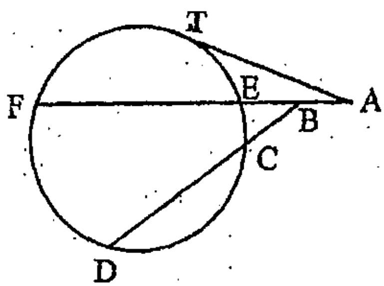
2. Find x in the following figures.



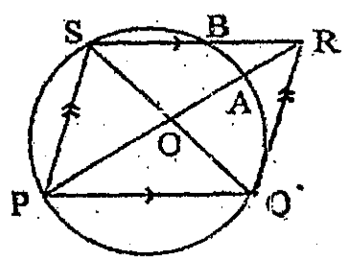
3. In the figure AC is tangent to the ΘABD; CBD and DAE are straight lines. Find BD and AE using the given data in the figure.



4. In the figure AT is a tangent segment; ABEF and BCD are straight lines,
(a) If AT = 6 cm , AB = BE = 2 cm , BC = 3 cm , then find EF and CD.
(b) If CD =8 cm ,BC=7cm ,BE= 2 cm , AB =4 cm , then find AT.

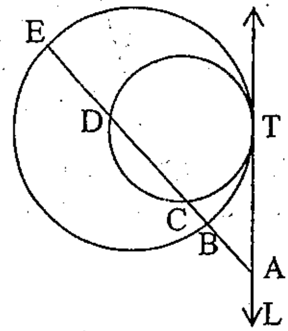


5. In parallelogram PQRS, PQ = 5 cm , PR =8 cm , QS = 6 cm.
Calculate the lengths of AR and BR.



6. ABCD is a square and E the middle point of CD. A circle drown through A, B and E meets BC at F. Prove CF = $\frac{1}{4}$ CB.

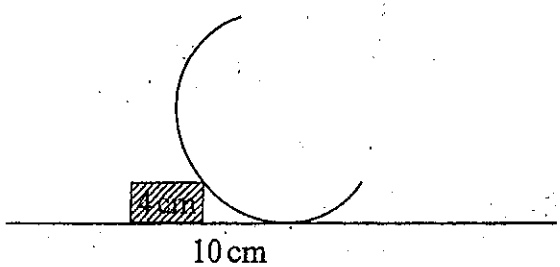
7. In the figure, A is any point of L except T, the common point of tangency of the circles. Prove that $\frac{AB}{AD} = \frac{AC}{AE}$.



8. M is the midpoint of a chord AB of a given circle; C is any point on the major arc AB and CM meets the circle at D. The circle tangent to AB at A and passes through C cuts CD at E. Prove that DM = ME.

9. Two circles intersect at A, B; X is any point on AB produced; a circle centre X, cuts one circle at P, Q and the second circle at L, M; XP, XM cut the circle PQA, LMA at S, T. Prove that PS = TM.

10. A brick 4 cm thick is placed so as to block a carriage wheel. If the distance of the brick from the point of contact of the wheel and the ground is 10 cm, find the radius of the wheel.



8.3 Concyclic Points and Converse Theorems

In the Grade 10 Text, we stated and proved a corollary that, no circle contains three different collinear points. In other words, any **three non collinear points lie on a circle.** Since any three non collinear points determine a triangle, a circle can be drawn through the vertices of a triangle.

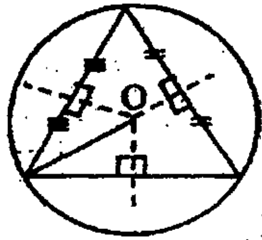


Fig. 8.16

Three of more points that lie on a circle are called "Concylic Points"

A triangle or quadrilateral is **cyclic** if there exists a circle that contains all of its vertices.

We can restate the above corollary for triangles as follows:

"Every triangle is cyclic"

Theorem 7 (Converse of Corollary 1.1)

If a straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

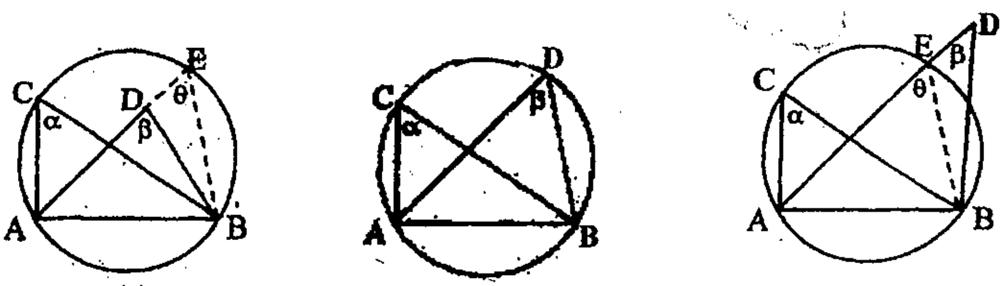


Fig. 8.17

Given: Straight line AB subtends the equal angles α and β at the points C and D on the same side of AB.

Prove: A, C, D, B are concyclic .

Proof: Let the circle through A,C,B be drawn.

If this $\odot$does not pass through D, then D must be either inside or outside the $\odot$.

If D lies inside $\odot$ACB [Fig.8.17(a)], produces AD to meet the circumference at E.

Join EB

$\alpha = \theta$ (in the same segment)

But $\alpha = \beta$; (given)

$\therefore \beta = \theta$,

which is impossible, since DB and EB are not parallel.

$\therefore$ D cannot lie inside ΘACB.

In the same way it can be shown that D cannot lie outside ΘACB [Fig.8.17 (c)].

Hence D must be on the circumference of ΘACB [Fig. 8.17 (b)];

$\therefore$ the points A,C,D,B are concyclic.

Theorem 8 (Converse of Corollary 1.2)

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

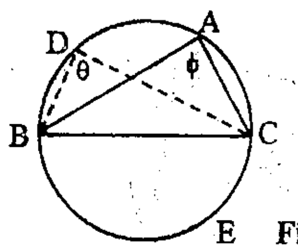


Fig. 8.18

Given: Right-angled Δ ABC with ΘBEC described on hypotenuse BC as diameter.

Prove : ΘBEC passes through vertex A.

Proof: Take any point D on the circumference and on the same side of BC as A. Join DB , DC .

θ is a rt. angle (angle in a semicircle)

ϕ is a rt. angle (given)

$\therefore \theta = \phi$

$\therefore$ B,D,A,C are concyclic (Equal angles subtend on the same side of BC)

$\therefore$ Θ BEC passes through vertex A.

Theorem 9 (Converse of Corollary 1.3)

If a pair of opposite angles of a quadrilateral are supplementary its vertices are concyclic.

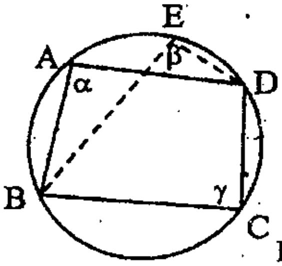


Fig. 8.19

Given: Quad. ABCD, in which α and γ are supplementary.

Prove: A, B, C, D are concyclic.

Proof: Let the $\odot$ through B, C, D be drawn. Take any point E on the circumference on the same side of BD as A.

Join EB, ED.

E, B, C, D are concyclic.

$\therefore$ β and γ are supplementary.

But α and γ are supplementary; (given)

$\therefore \alpha = \beta$

$\therefore$ B, A, E, D are concyclic (Th. 7)

$\therefore$ A lies on the arc BED, i.e, on the $\odot$ through B, C, D;

$\therefore$ A, B, C, D are concyclic.

Theorem 10 (Converse of Th.5 and Corollary 6.1)

If two line segments, AB and CD intersect at a point P internally or externally, such that AP . PB = CP . PD, the four points A, B, C, D are concyclic.

Example 1.

From the figure with respective given measures of angles, prove that BCEF is a cyclic quadrilateral.

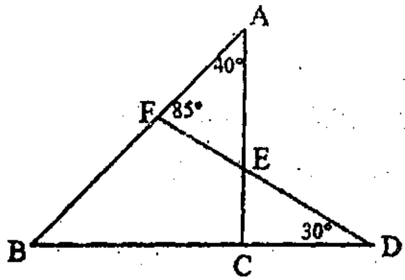


Fig. 8.20

Proof: $\angle CED = \angle AEF = 180° - (85° + 40°) = 55°$

$\angle BCE = 55° + 30° = 85° = \angle AFE$

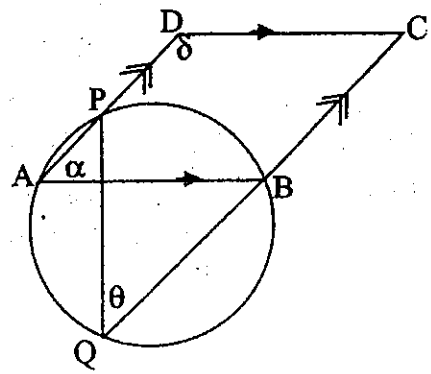
But $\angle BFE + \angle AFE = 180°$

$\therefore \angle BFE + \angle BCE = 180°$

Hence BCEF is cyclic.

Example 2.

ABCD is a parallelogram.
Any circle through A and B cuts DA and CB at P and Q as shown.
Prove that DCQP is cyclic.



Proof: $\alpha = \theta$

(Subtends on the same arc PB)

Fig. 8.21

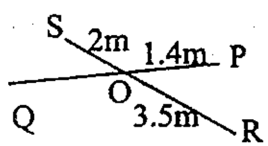
Since ABCD is a ||gm (parallelogram), $\alpha + \delta = 180°$

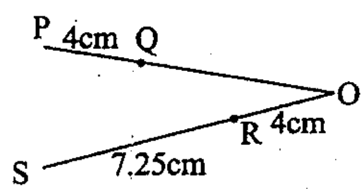
$\therefore \quad \theta + \delta = 180°$

Hence DCQP is cyclic.

Exercise 8.3

1. When will the points P, Q, R, S be concyclic?

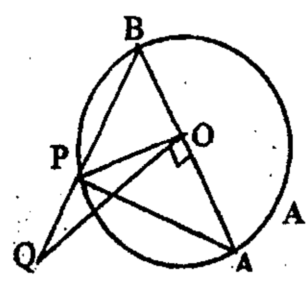




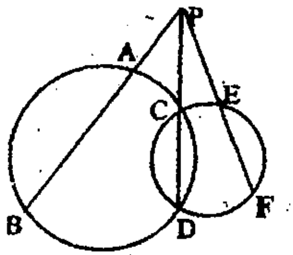
2. Given: ⊙ O

Prove: (a) A,Q,P,O are concyclic.

(b) $\angle OPA = \angle OQB$.

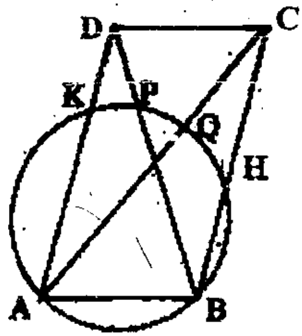


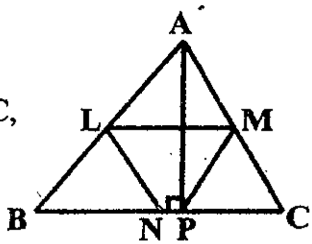
3. Prove: A,B,E,F are concyclic.



4. Given: ABCD is a parallelogram; A circle through A, B cuts BC, AC, BD and AD at H, Q, P, K.

Prove: (a) C, D, P, Q are concyclic.
(b) C,D, H, K are concyclic.





If L, M, N are the middle points of the sides of Δ ABC, and P is the foot of perpendicular from A to BC. Prove that L, N, P, M are concyclic.

If L, M, N be the middle points of the sides of a triangle, and if P, Q, R be the feet of the perpendiculars from the vertices on the opposite sides, prove P, N, Q, L, M, R are concyclic.

ABC is a triangle inscribed in a circle and DE the tangent at A. A line drawn parallel to DE meets AB, AC at F, G respectively. Prove BFGC is a cyclic quadrilateral.

Two circles cut at A, B and through A any line CAD is drawn meet the circles at C, D. CB and DB are joined and produced to meet the circles again at E, F. If CF, DE produced meet at G, prove the points B, F, G, E are concyclic.

9. Two incongruent circles P and Q intersect at A and D, a line BDC is drawn to cut the circle P at B and circle Q at C, and such that $\angle BAC = 90°$. Prove that APDQ is cyclic.

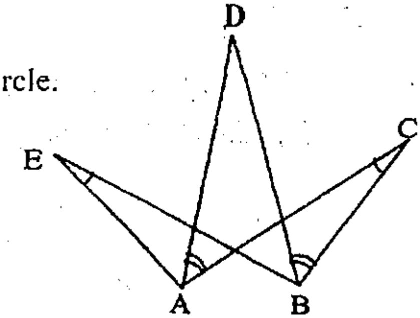
10. ABC is a triangle in which AB = AC. P is a point inside the triangle such that $\angle PAB = \angle PBC$. Q is the point on BP produced such that AQ = AP. Prove that ABCQ is cyclic.

11. Two circles intersect at A and B. A point P is taken on one so that PA and PB cut the other at Q and R respectively. The tangents at Q and R meet the tangent at P in S and T respectively. Prove that

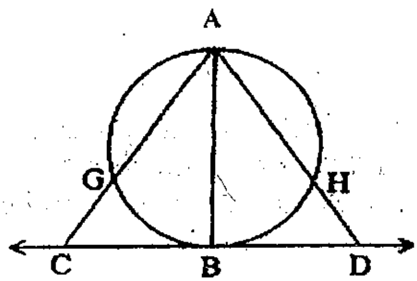
    (a) $\angle TPR = \angle BRQ$

    (b) PBQS is cyclic.

12. Prove: A,B,C,D and E all lie on one circle.

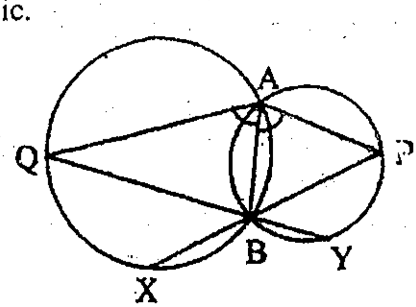


13. In the figure, AB is a diameter and CD is the tangent at B.
Prove that AC . AG = AD . AH.



14. In ΔABC, AB = AC. P is any point on BC, and Y any point on AP. The circles BPY and CPY cut AB and AC respectively at X and Z. Prove XZ//BC.

15. In the figure, PBX and QBY are segments and $\angle PAB = \angle QAB$. Prove that PQXY is cyclic.



16. Prove that the quadrilateral formed by producing the bisectors of the interior angles of any quadrilateral is cyclic.

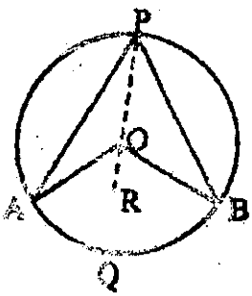
17. ABC is a triangle, in which AX, BY, CZ are the perpendiculars from the vertices to the opposite sides. If the perpendiculars meet at O, prove that AO.OX = BO.OY = CO . OZ.

18. AB is a diameter of a circle and E any point on the circumference. From any point C on AB produced, a line is drawn perpendicular AB, meeting AE produced at D. Prove that AE. AD = AB . AC.

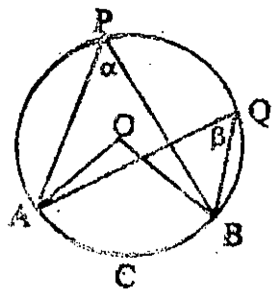
19. From any point D on the base BC of Δ ABC a line is drawn meeting AB at E and such that $\angle$ BDE = $\angle$A. Prove BE . BA = BD . BC.

SUMMARY

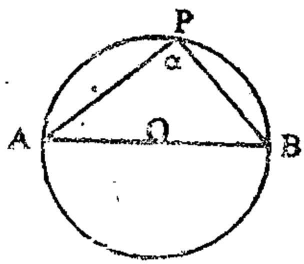
Angles in a Circle



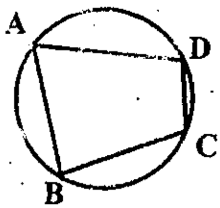
$\angle$**AOB = 2**$\angle$**APB.**



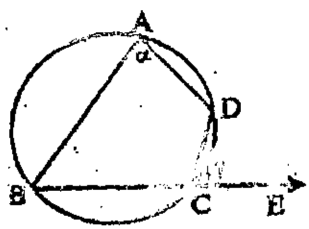
$\alpha = \beta$



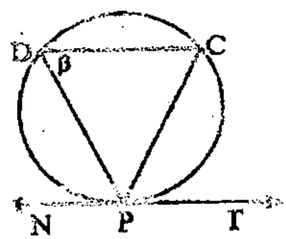
α = 1 rt. $\angle$



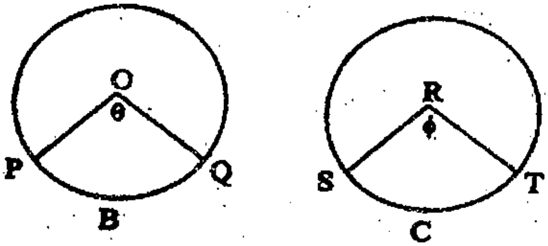
$\angle$**A +** $\angle$**C= 2 rt.** $\angle$**s**



$\gamma = \alpha$

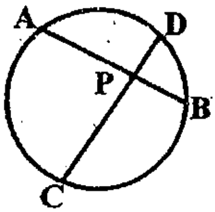


$\angle$ **CPT =** β

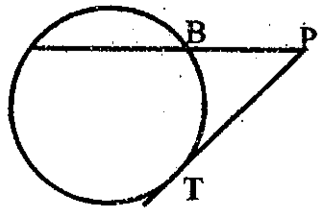


$\theta = \phi \iff$ arc PBQ = arc SCT

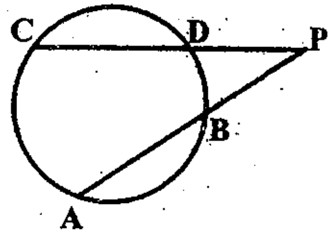
Product Properties of Circles



AP . PB = CP . PD

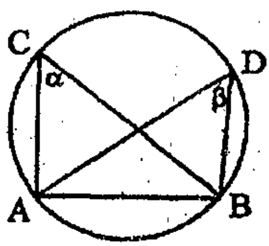


PA. PB = PT^2

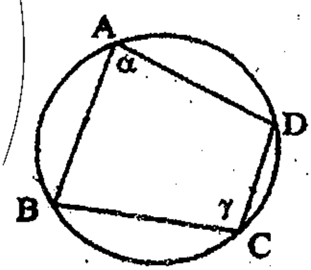


PA . PB = PC . PD

Concyclic Points and Converse Theorems



$\alpha = \beta \Rightarrow$ A,C,D, B are concyclic.



$\alpha + \gamma = 180° \Leftrightarrow$ A,B,C,D are concyclic.

CHAPTER 9

Areas of Similar Triangles

9.1 Areas of Similar Triangles

We have learned in the Grade 10. Text the relationship of the sides, angles and other segments of similar triangles. Now we shall compare the areas of similar triangles.

Theorem 1

The areas of two similar triangles have the same ratio as the squares of the lengths of any two corresponding sides.

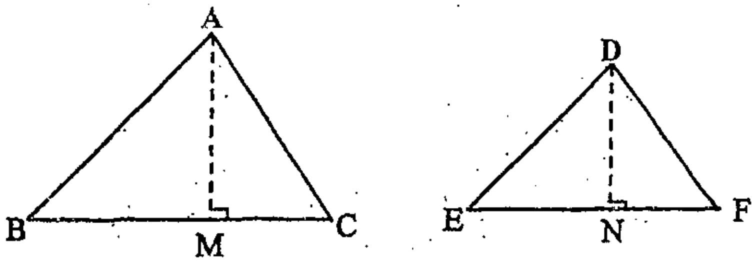


Fig. 9.1

Given : Similar Δ s ABC, DEF in which BC, EF are corresponding sides.

Prove : $\alpha\,(\Delta\,ABC) : \alpha\,(\Delta\,DEF) = BC^2 : EF^2$

Proof: Draw $AM \perp BC$ and $DN \perp EF$.

$$\frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{\frac{1}{2}BC.AM}{\frac{1}{2}EF.DN} = \frac{BC}{EF}.\frac{AM}{DN}$$

But $\Delta\,ABM \sim \Delta\,DEN$ $(\because \angle B = \angle E,\ \angle AMB = \angle DNE)$

$$\therefore \frac{AM}{DN} = \frac{AB}{DE} = \frac{BC}{EF}$$

$$\therefore \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{BC}{EF}.\frac{BC}{EF} = \frac{BC^2}{EF^2}$$

i.e., $\alpha\,(\Delta\ ABC) : \alpha\,(\Delta\ DEF) = BC^2 : EF^2$

Corollary 1.1

The ratio of the areas of two similar triangles equals to the ratio of the squares of any two corresponding altitudes.

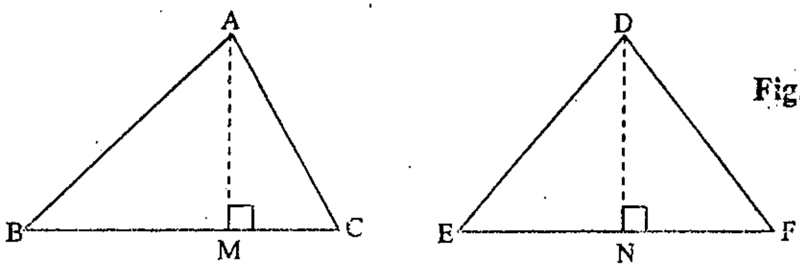


Fig. 9.2

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AM^2}{DN^2}$$

Corollary 1.2

The ratio of the areas of two similar triangles equals to the ratio of the squares of any two corresponding medians.

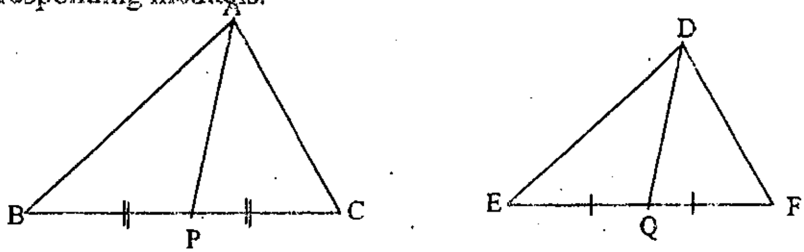


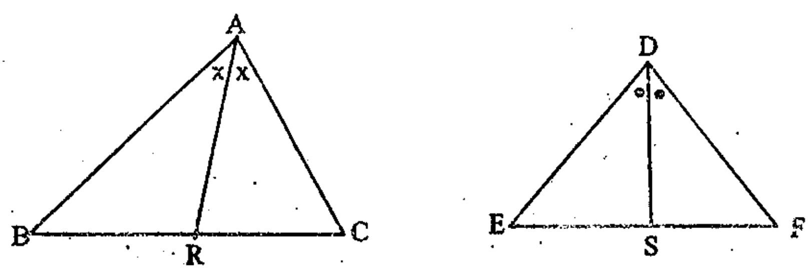
Fig. 9.3

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AP^2}{DQ^2}$$

Corollary 1.3

The ratio of the areas of two similar triangles equals to the ratio of the squares of any two corresponding angle bisectors.

Fig. 9.4



$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AR^2}{DS^2}$$

Example 1. The areas of two similar triangles are 56.25 sq.cm. and 42.25 sq. cm. respectively. Find the ratio of their altitudes.

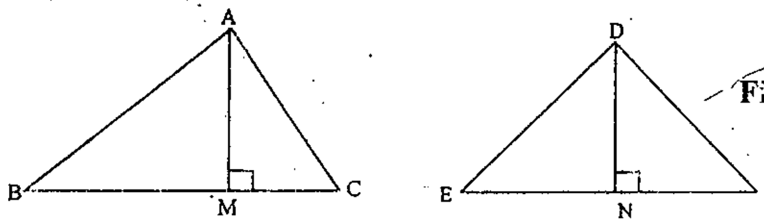


Fig. 9.5

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AM^2}{DN^2}$$

$$\frac{56.25}{42.25} = \frac{AM^2}{DN^2}$$

$$\frac{225}{169} = \frac{AM^2}{DN^2}$$

$$\frac{AM}{DN} = \frac{15}{13}$$

i.e., AM:DN =15:13

Example 2. ΔABC is an isosceles right triangle with $\angle A$ the right angle E and D are points on opposite side of AC, with E on the same side of AC as B, such that ΔACD and ΔBCE are both equilateral. Prove that $\alpha(\Delta BCE) = 2\alpha(\Delta ACD)$.

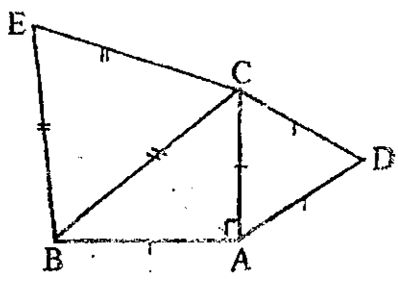


Fig. 9.6

Given: Δ ABC is an isosceles right triangle with rt. $\angle$ at A.

Δ ADC and Δ BCE are equilateral Δ s

Prove: $\alpha(\Delta BCE) = 2\alpha(\Delta ACD)$

Proof: Since Δ BCE and Δ ADC are equilateral, they are similar.

$$\therefore \frac{\alpha(\Delta BCE)}{\alpha(\Delta ACD)} = \frac{BC^2}{AC^2}$$

In isosceles rt. Δ BAC, $BC = \sqrt{2}AC$.

$$\frac{\alpha(\Delta BCE)}{\alpha(\Delta ACD)} = \frac{(\sqrt{2}AC)^2}{AC^2} = 2$$

Hence $\alpha(\Delta BCE) = 2\alpha(\Delta ACD)$.

Now we will state some results which can be proved easily and are useful.

Statement I. The ratio of the areas of two parallelograms having equal bases (altitudes) equals to the ratio of the corresponding altitudes (bases).

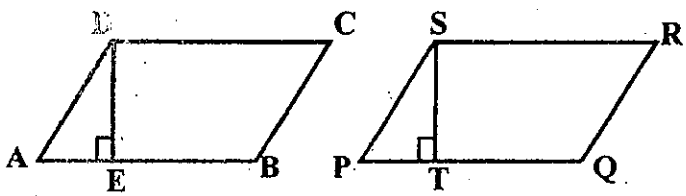


Fig. 9.7

In parallelograms ABCD and PQRS, if base AB = PQ, then

$$\frac{\alpha(ABCD)}{\alpha(PQRS)} = \frac{DE}{ST}$$

In parallelograms ABCD and PQRS, if altitude DE = ST, then

$$\frac{\alpha(ABCD)}{\alpha(PQRS)} = \frac{AB}{PQ}$$

Statement II. The ratio of the areas of two triangles having equal bases (altitudes) equals the ratio of the corresponding altitudes (bases).

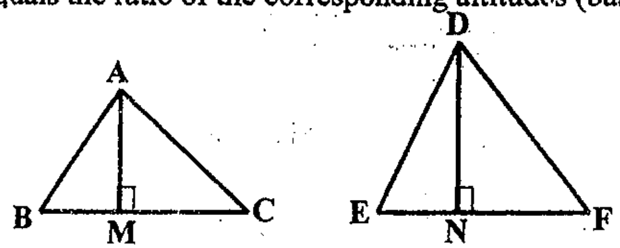


Fig.9.8

In ΔABC and ΔDEF, if bases $BC = EF$, then $\frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AM}{DN}$

In Δ ABC and Δ DEF, if altitudes $AM = DN$, then $\frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{BC}{EF}$

Example 3. In a given circle PA is a tangent segment and PBC is a secant segment

Prove that $\frac{AB^2}{CA^2} = \frac{PB}{PC}$

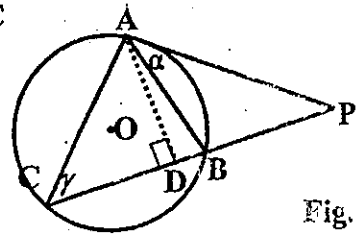


Fig. 9.9

Given: In ⊙O, PA is a tangent segment and PBC is a secant segment.

Prove: $\frac{AB^2}{CA^2} = \frac{PB}{PC}$

Proof: Draw $AD \perp PC$.

In Δ PAB and Δ PCA

$\alpha = \gamma$ (angle between tangent & chord = angle in the alternate segment)

$\angle P = \angle P$ (common $\angle$)

$\Delta PAB \sim \Delta PCA$ (AA Cor:)

$$\frac{\alpha(\Delta PAB)}{\alpha(\Delta PCA)} = \frac{AB^2}{CA^2}$$

But Δ PAB and Δ PCA have the same altitude AD.

$$\frac{\alpha(\Delta PAB)}{\alpha(\Delta PCA)} = \frac{PB}{PC}$$

$$\therefore \frac{AB^2}{CA^2} = \frac{PB}{PC}$$

Exercise 9.1

1. The bases of two similar triangles are 2.5 cm and 3.5 cm, respectively. The area of the smaller triangle is 3.75 sq. cm. Find the area of the larger triangle.

2. A straight line drawn parallel to the base BC of Δ ABC cuts the sides AB, AC in the ratio 2:3. Find the area of the triangle thus cut off, if the area of the whole triangle be 72.25 sq. cm.

3. Δ ABC is bisected by a line PQ drawn parallel to its base BC. In what ratio does PQ divide the sides of the triangle?

4. In trapezium ABCD. AB is twice DC and AB//DC. If AC, BD, intersect at O, prove that $\alpha(\Delta AOB) = 4\alpha(\Delta COD)$.

5. Two chords AC, BD of a circle intersect at O. Prove that $\alpha(\Delta AOB):\alpha(\Delta COD) = OA^2:OD^2$.

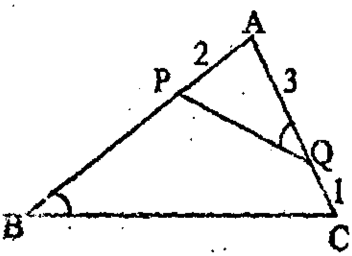
6. ABC is a triangle, and BE, CF are the perpendiculars drawn to the sides AC, AB. Prove that $\alpha(\Delta ABE):\alpha(\Delta ACF) = AB^2:AC^2$.

7. PA and PB are the tangent segments at A and B to a circle whose centre is O. Prove that $\alpha(\Delta PAB):\alpha(\Delta OAB) = AP^2:AO^2$

8. In the figure $\angle AQP = \angle B$. Find the length of PB and the ratios

(a) $\alpha(\Delta APQ):\alpha(\Delta ABC)$

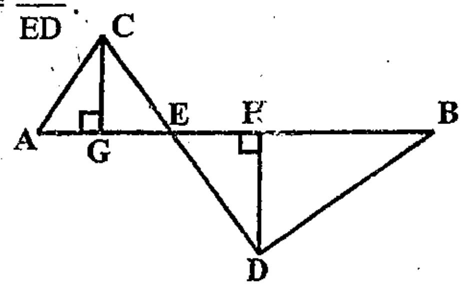
(b) $\alpha(\Delta APQ):\alpha(BCQP)$



9. In ΔABC, D is a point of AC such that AD = 2CD. E is on BC such that DE//AB. Compare the areas of ΔCDE and Δ ABC. If $\alpha(ABED) = 40$, what is $\alpha(\Delta ABC)$?

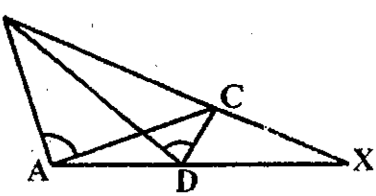
10. In the figure, CG $\perp$ AE, DH $\perp$ BE and $\dfrac{EA}{EC} = \dfrac{EB}{ED}$.

If CG = 6, DH = 8 and AB = 35.



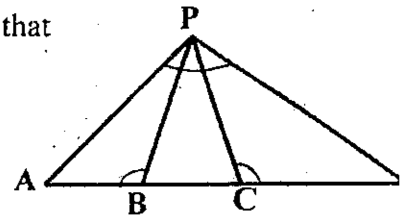
Find (a) $\dfrac{\alpha(\Delta ACE)}{\alpha(\Delta BDE)}$ (b) $\alpha(\Delta BDE)$

11. ADX and BCX are two segments such that $\angle BAC = \angle BDC$. Prove that $\dfrac{\alpha(\Delta ABX)}{\alpha(\Delta CDX)} = \dfrac{AB^2}{CD^2}$.



12. ABCD is a segment and P a point outside it such that $\angle PBA = \angle PCD = \angle APD$. Prove that

$\dfrac{\alpha(\Delta ABP)}{\alpha(\Delta PCD)} = \dfrac{AB^2}{BP^2}$.



13. ABCD is a trapezium in which AB//CD and $\angle ADB = \angle C$. Prove that $AD^2 : BC^2 = AB : CD$.

14. In ΔPQR, $\angle P = 90°$ and $PS \perp QR$. If $QR = 3PQ$, prove that $SR = 8QS$.

15. ABC is a triangle such that BC : CA : AB = 3 : 4 : 5. If BPC, CQA, ARB are equilateral triangles, prove that $\alpha(\Delta BPC) + \alpha(\Delta CQA) = \alpha(\Delta ARB)$.

16. ΔABC is inscribed in a circle. Straight lines are drawn through B and C parallel to CA and BA respectively, to meet the tangent at A in D and E. Prove that

$$\frac{DA}{AE} = \frac{AB}{EC} = \frac{AB^2}{AC^2}.$$

17. In ΔABC, AD and BE are the altitudes. If $\alpha(\Delta DEC) = \frac{3}{4}\alpha(\Delta ABC)$, prove that $\angle ACB = 30°$.

18. A, B, C, D are four points in order on a circle O, so that AB is a diameter and $\angle COD = 90°$. AD produced and BC produced meet at E. Prove that $\alpha(\Delta ECD) = \alpha(ABCD)$.

19 ABC, AD and BE are altitudes. If $\angle ACB = 45°$, prove that $\alpha(\Delta DEC) = \alpha(ABDE)$.

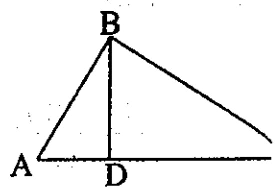
20. A, B, C, D are four points in order on a circle O, so that AB is a diameter. AD produced and BC produced meet at E. If $\alpha(\Delta ECD) = \alpha(ABCD)$, prove that $DC = \sqrt{2}\,AO$.

21. In ΔPQR, QR = 16cm. The point Y on PR is such that PY = 3cm, YR = 5cm. The point X on PQ is such that XY//QR. Find the length in cm of XY. If $\alpha(\Delta PXY) = 6\ cm^2$, then find $\alpha(QXYR)$.

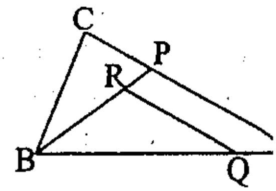
22. ABC is a right triangle with A the right angle. E and D are points on opposite side of AC, with E on the same side of AC as B, such that ΔACD and ΔBCE are both equilateral. If $\alpha(\Delta BCE) = 2\ \alpha(\Delta ACD)$, prove that ABC is an isosceles right triangle.

23. The chords XB and AY of a circle intersect at S. If XS = 4cm, SA = 5cm, then prove that $\Delta XYS \sim \Delta ABC$, and hence, find $\alpha(\Delta XYS) : \alpha(\Delta ABS)$.

24. In the figure AB = 6cm, AC = 9cm, and D is a point on AC such that $\angle ABD = \angle ACB$. Calculate AD. Given that $\alpha(\Delta ABD) = 10cm^2$, calculate $\alpha(\Delta ABC)$.

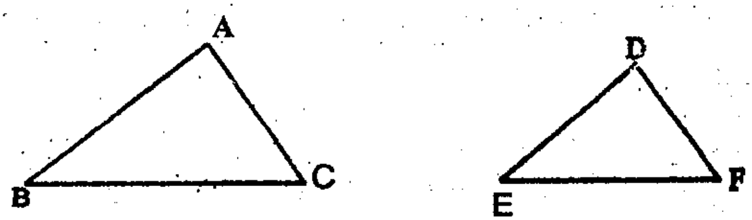


25. In the diagram, P is the point on AC, such that AP = 2 PC, R is the point on BP such that BR = 3 RP and QR // AC. Given that $\alpha(\Delta BPA) = 32\ cm^2$, calculate $\alpha(\Delta BPC)$, $\alpha(\Delta BRQ)$.

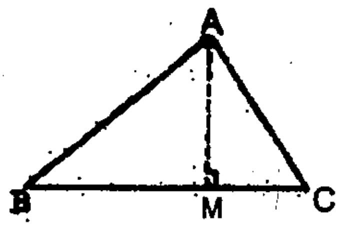


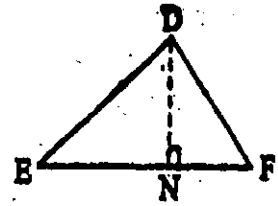
SUMMARY

Areas of Similar Triangles

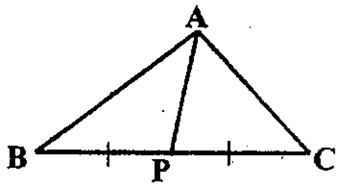


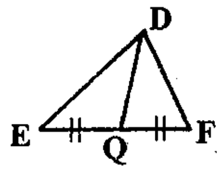
$\Delta\ ABC \sim \Delta\ DEF \Rightarrow \quad \alpha\ (\Delta\ ABC) : \alpha\ (\Delta\ DEF) = BC^2 : EF^2$



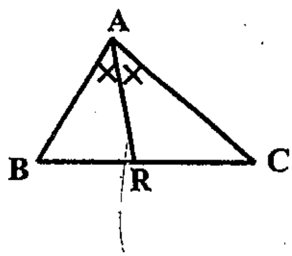


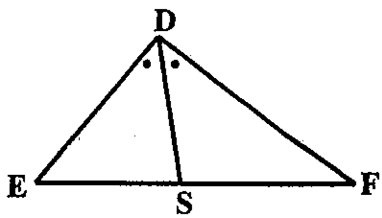
$$\Delta\ ABC \sim \Delta\ DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AM^2}{DN^2}.$$





$$\Delta\ ABC \sim \Delta\ DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AP^2}{DQ^2}$$





$$\Delta\ ABC \sim \Delta\ DEF \Rightarrow \frac{\alpha(\Delta ABC)}{\alpha(\Delta DEF)} = \frac{AR^2}{DS^2}$$

CHAPTER 10

Introduction to Vectors and Transformation Geometry

10.1 Geometric Vectors

A quantity that has only magnitude is called a **scalar quantity**. Examples of scalar quantities are mass, time, energy, length, area, density and volume. A quantity that possesses both magnitude and direction is called **a vector quantity**. Examples of vector quantities are displacement, velocity, acceleration, force, pressure and momentum.

A vector quantity can be presented by a line segment with the direction specified. It is called **a geometric vector.** For example, the directed line segment in Fig. 10.1 represents a vector. The symbol $\overrightarrow{AB}$ is used to denote the vector. The point A is called the **initial point** and the point B is called the **terminal point.**

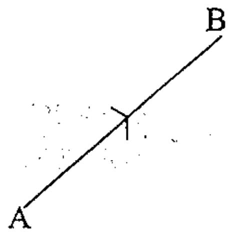


Fig. 10.1

Vectors will also be denoted by small letters $\vec{a}$, $\vec{b}$, $\vec{c}$, ...

Definition 1

A geometric vector $\vec{a}$ is a line segment with a specified direction. The magnitude of $\vec{a}$ is the length of $\vec{a}$ and denoted by $|\vec{a}|$, and this is read as the modulus of $\vec{a}$.

The magnitude of $\overrightarrow{AB}$ is also written as $|\overrightarrow{AB}|$ or simply as **AB**.

Definition 2

Two geometric vectors are said to be equal if they have the same magnitude and the same direction. The symbol ' = ' will be used to indicate this equal relationship.

For example, if ABCD is a parallelogram, then the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal and the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are also equal.

We write as $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$.

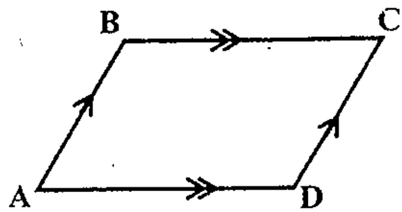


Fig. 10.2

Definition 3

A geometric vector which has magnitude equal to zero is called a **zero vector** and denoted by $\overrightarrow{0}$.

Definition 4

A geometric vector having the same magnitude as $\vec{a}$ but a direction opposite to that of $\vec{a}$ is called the **negative** of $\vec{a}$ and denoted by $-\vec{a}$.

Note that $\overrightarrow{AB} = -\overrightarrow{BA}$.

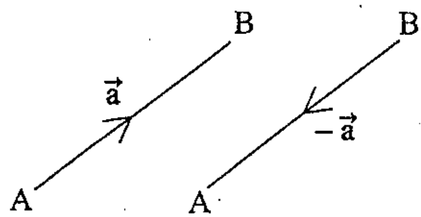


Fig. 10.3

Definition 5

For geometric vectors $\vec{a}$ and $\vec{b}$, $\vec{a}+\vec{b}$ is the geometric vector having as its initial point the initial point of $\vec{a}$ and as its terminal point the terminal point of $\vec{b}$, where the terminal point of $\vec{a}$ is the initial point of $\vec{b}$. The geometric vector $\vec{a}+\vec{b}$ is called the **sum** of $\vec{a}$ and $\vec{b}$.

If $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$, then $\vec{a}+\vec{b} = \overrightarrow{AC}$. See Fig. 10.4. Since a triangle is formed by $\vec{a}$, $\vec{b}$ and $\vec{a}+\vec{b}$ in Fig. 10.4 (a), we say that geometric vectors are added according to " **The Triangle Rule : $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.**"

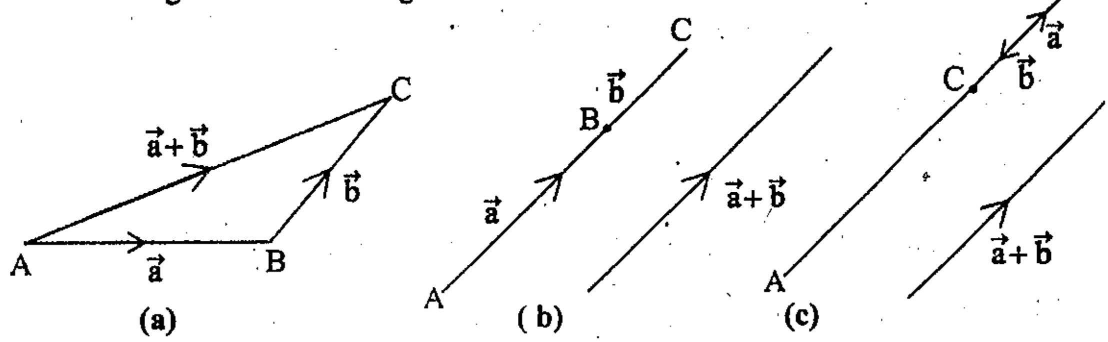
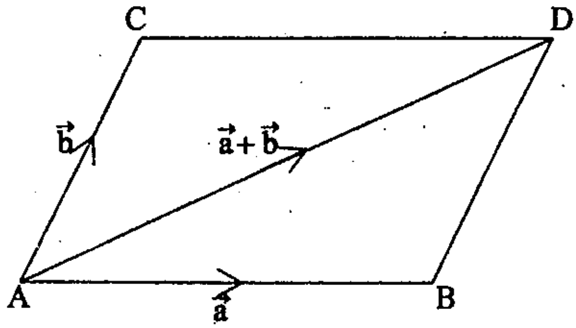


Fig. 10.4

Fig. 10.4 (b) and Fig. 10.4 (c) show the sum $\vec{a}+\vec{b}$, if A , B and C are collinear.

Note that $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$.

om the properties of a parallelogram we have that if $\vec{a}$ and $\vec{b}$ have the same initial point, the sum $\vec{a}+\vec{b}$ is a diagonal of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$. If $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$, then $\vec{a}+\vec{b} = \overrightarrow{AD}$ where ABDC is the parallelogram. See Fig. 10.5. Therefore we also say that geometric vectors are added according to " **The Parallelogram Rule : $\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AD}$** ".



$$\vec{a}+\vec{b} = \overrightarrow{AB}+\overrightarrow{AC}$$
$$= \overrightarrow{AB}+\overrightarrow{BD}$$
$$= \overrightarrow{AD}$$

Fig.10.5

Now consider the addition of three geometric vectors $\vec{a}$, $\vec{b}$, $\vec{c}$.

Let $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$ and $\overrightarrow{CD}=\vec{c}$. We have

$$\vec{a}+\vec{b} = \overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$$

and consequently

$$\vec{a}+\vec{b}+\vec{c} = \overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}.$$

Thus the vector sum $\vec{a}+\vec{b}+\vec{c}$ is the vector obtained by joining the initial point of $\vec{a}$ to the terminal point of $\vec{c}$ as shown in Fig. 10.6.

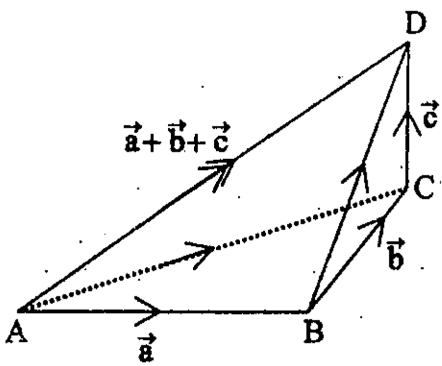


Fig.10.6

Alternatively,

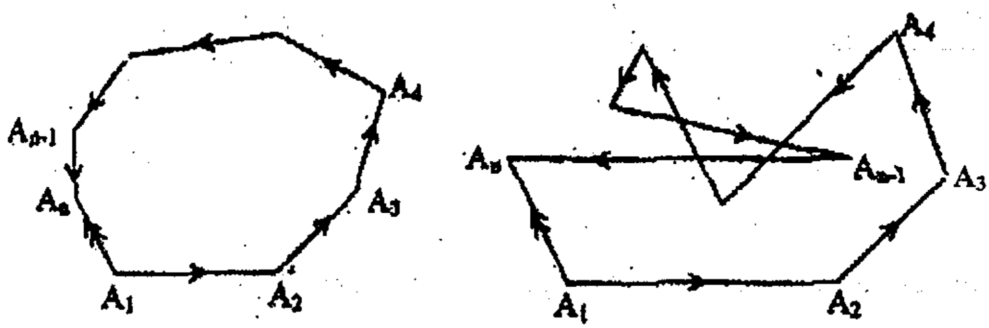
$$\vec{b}+\vec{c}=\overrightarrow{BC}+\overrightarrow{CD}$$
$$=\overrightarrow{BD},$$
$$\vec{a}+\vec{b}+\vec{c}=\overrightarrow{AB}+\overrightarrow{BD}$$
$$=\overrightarrow{AD}.$$

In the same manner, we can deduce the following rule.

Polygon Rule of Vector Addition

If $A_1A_2A_3 \ldots A_n$ is a closed polygon, then

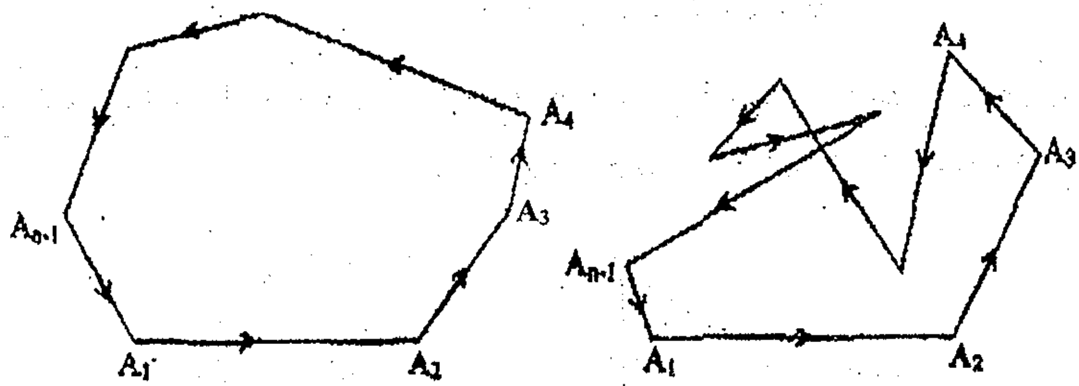
$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\overrightarrow{A_3A_4}+\cdots+\overrightarrow{A_{n-1}A_n}=\overrightarrow{A_1A_n}.$$



(a) Fig. 10.7 (b)

If A_1 and A_n coincide, then

$$\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\overrightarrow{A_3A_4}+\cdots+\overrightarrow{A_{n-1}A_1}=\vec{0}.$$



(a) Fig. 10.8 (b)

Example 1.

ABCDEF is a regular hexagon. If $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AF} = \vec{a}$, find $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$ and $\overrightarrow{EF}$ in terms of $\vec{a}$ and $\vec{b}$.

[All triangles formed by joining the common point of intersection of the diagonals to the vertices of the hexagon are equilateral , and opposite sides of the hexagon are parallel.]

Solution

Let G be the common point of intersection of the diagonals.

$\overrightarrow{BC} = \overrightarrow{AG} = \vec{a} + \vec{b}$

$\overrightarrow{CD} = \overrightarrow{AF} = \vec{a}$

$\overrightarrow{DE} = \overrightarrow{BA} = -\overrightarrow{AB} = -\vec{b}$

$\overrightarrow{EF} = \overrightarrow{CB} = -\overrightarrow{BC} = -(\vec{a} + \vec{b})$

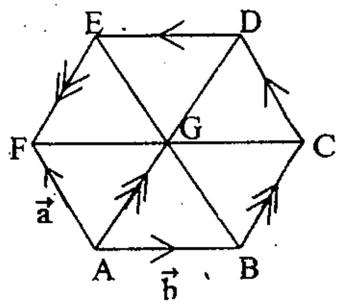


Fig.10.9

Definition 6

For geometric vectors $\vec{a}$ and $\vec{b}$, $\vec{a} - \vec{b}$ is the geometric vector defined by $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

The geometric vector $\vec{a} - \vec{b}$ is called the **difference** of $\vec{a}$ and $\vec{b}$.

The construction of $\vec{a} - \vec{b}$ is shown in two ways **in Fig.** 10.10.

If $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$, then

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \overrightarrow{AB} + (-\overrightarrow{AC}) = \overrightarrow{AB} + \overrightarrow{CA} = \overrightarrow{CB}$. See Fig. 10.10

(a). We can also draw $\overrightarrow{AD}$ to represent $-\vec{b}$. Then

$$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$$, See Fig. 10.10(b).

Observe that $\overrightarrow{CB} = \overrightarrow{AE}$.

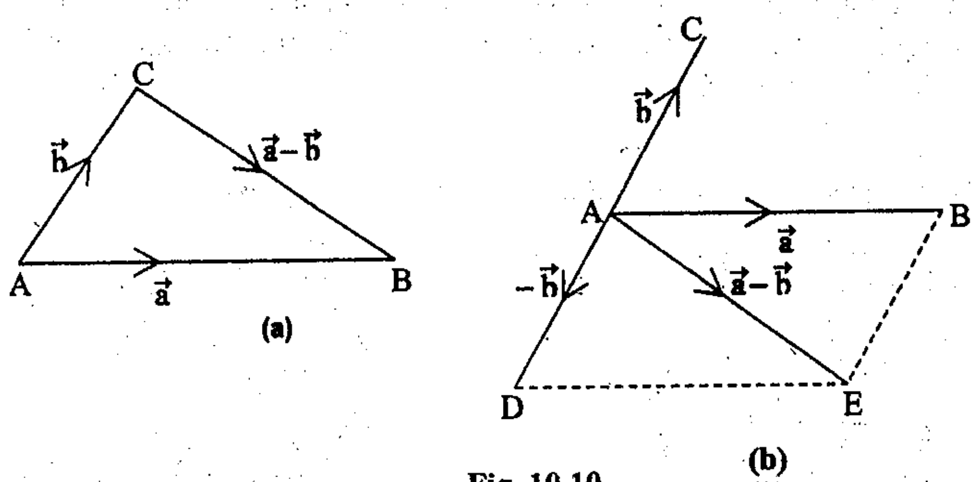


Fig. 10.10

From Fig. 10.10, we can see that $\vec{b} + (\vec{a} - \vec{b}) = \vec{a}$.

Definition 7

The product of a geometric vector $\vec{a}$ by a scalar k, denoted by $k\vec{a}$, is a geometric vector whose magnitude is $|k|$ times that of $\vec{a}$, and whose direction is the same, or opposite to that of $\vec{a}$, according as k is positive or negative.

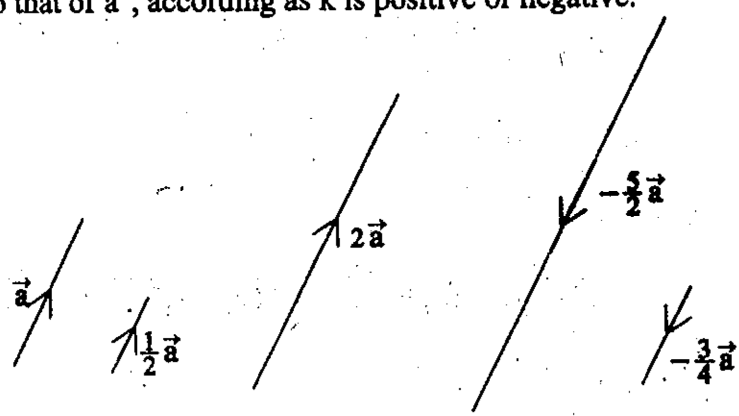


Fig. 10.11

Note that $1\vec{a} = \vec{a}$, $0\vec{a} = \vec{0}$ and $(-1)\vec{a} = -\vec{a}$.

The scalar multiplication obeys the following laws.

If k_1, k_2 are scalars and, $\vec{a}$, $\vec{b}$ are geometric vectors, then

(i) $k_1(k_2\vec{a}) = (k_1k_2)\vec{a}$ (Associative Law)

(ii) $(k_1 + k_2)\vec{a} = k_1\vec{a} + k_2\vec{a}$ (Distributive Law)

(iii) $k_1(\vec{a} + \vec{b}) = k_1\vec{a} + k_1\vec{b}$ (Distributive Law)

Two geometric vectors are said to be parallel if they have the same direction or opposite directions regardless of whether or not they have the same magnitude. Thus $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} = k\vec{b}$, where k is a scalar. If k is positive, $\vec{a}$ and $\vec{b}$ are in the same direction. If k is negative, $\vec{a}$ and $\vec{b}$ **are in opposite directions.**

Consequently, **it can be seen** that the points A,B and C are collinear if and only if $\overrightarrow{AB} = k\overrightarrow{BC}$, where k is a non-zero scalar.

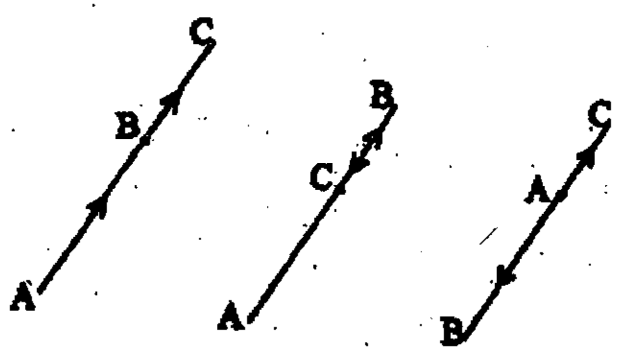


Fig. 10.12

Example 2.

OPRQ is a parallelogram and OP is produced to S such that $\overrightarrow{OS} = 3\overrightarrow{OP}$.

If X is a point on PR such that $\overrightarrow{PX} = 2\overrightarrow{XR}$, show that the points Q, X and S are collinear.

Solution

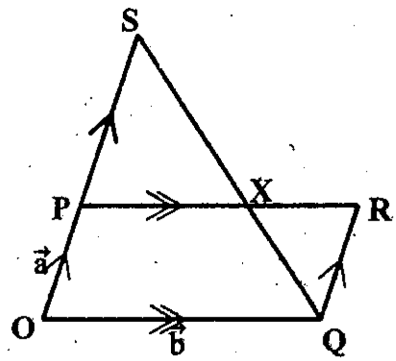


Fig. 10.13

Let $\overrightarrow{OP} = \vec{a}$ and $\overrightarrow{OQ} = \vec{b}$.

Then $\overrightarrow{OS} = 3\vec{a}$ and $\overrightarrow{PS} = 2\vec{a}$.

Since $\overrightarrow{PR} = \vec{b}$, we get

$$\overrightarrow{PX} = \frac{2}{3}\vec{b},\ \overrightarrow{XR} = \frac{1}{3}\vec{b}.$$

Now, we have

$$\overrightarrow{SX} = \overrightarrow{SP} + \overrightarrow{PX} = -2\vec{a} + \frac{2}{3}\vec{b}$$

and

$$\overrightarrow{XQ} = \overrightarrow{XR} + \overrightarrow{RQ} = \frac{1}{3}\vec{b} - \vec{a}.$$

Thus

$$\overrightarrow{SX} = 2(-\vec{a} + \frac{1}{3}\vec{b}) = 2\overrightarrow{XQ}$$

Therefore Q, X and S are collinear.

Theorem 1

Let $\vec{a}$ and $\vec{b}$ be non-zero and non-parallel vectors. If $h\vec{a} = k\vec{b}$ then $h = k = 0$.

Proof

Suppose that $h \neq 0$.

Then $\vec{a} = \frac{k}{h}\vec{b}$.

Hence $\vec{a}$ and $\vec{b}$ are parallel.

This contradicts to the hypothesis of the theorem.

Thus h must be zero.

Then $k\vec{b} = h\vec{a} = 0\vec{a} = \vec{0}$.

Since $\vec{b} \neq \vec{0}$, k must be zero.

Therefore $h = k = 0$.

Corollary 1.1

Let $\vec{a}$ and $\vec{b}$ be non-zero and non-parallel vectors.

If $h\vec{a} + k\vec{b} = m\vec{a} + n\vec{b}$ then $h = m$ and $k = n$.

Proof

If $h\vec{a} + k\vec{b} = m\vec{a} + n\vec{b}$, we get

$$(h - m)\ \vec{a} = (n - k)\ \vec{b}.$$

By Theorem 1,

$$h - m = 0 \text{ and } n - k = 0$$

which give $h = m$ and $n = k$.

Example 3.

It is given that $\vec{u} = 5\vec{a} + 4\vec{b}$, $\vec{v} = 3\vec{a} - \vec{b}$ and $\vec{w} = (2h - k)\ \vec{a} + (h + k + 3)\ \vec{b}$, where $\vec{a}$ and $\vec{b}$ are not parallel. If $\vec{w} = 2\vec{u} - 3\vec{v}$, calculate the value of h and of k. ($\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$)

Solution $\quad \vec{w} = 2\vec{u} - 3\vec{v}$

$$= 2(5\vec{a} + 4\vec{b}) - 3(3\vec{a} - \vec{b})$$

$$(2h - k)\ \vec{a} + (h + k + 3)\ \vec{b} = \vec{a} + 11\ \vec{b}$$

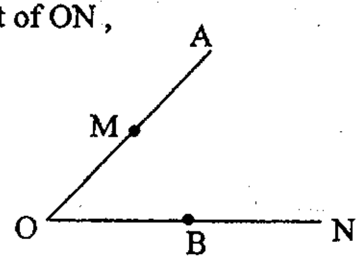
We have $\quad 2h - k = 1$ and $h + k + 3 = 11$.

Solving these equations, we get

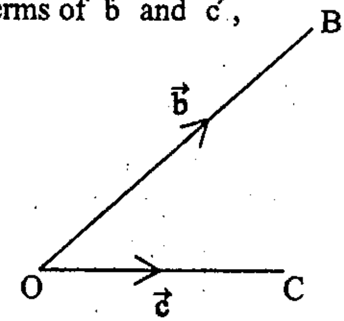
$$h = 3 \text{ and } k = 5.$$

Exercise 10.1

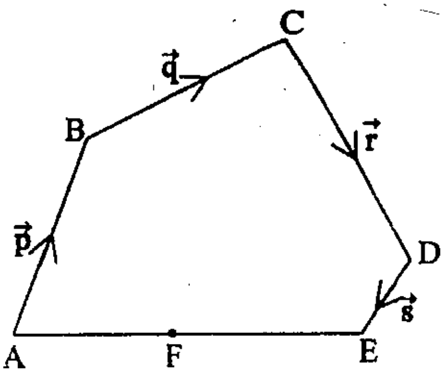
1. In the figure $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the vectors $\vec{a}$ and $\vec{b}$ respectively. If M is the midpoint of OA and B is the midpoint of ON, write down $\overrightarrow{BM}$ and $\overrightarrow{MN}$ in terms of $\vec{a}$ and $\vec{b}$.



2. In the figure $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Make the points E and F such that $\overrightarrow{OE} = \frac{1}{2}\vec{b}$, $\overrightarrow{OF} = -2\vec{c}$. Find, in terms of $\vec{b}$ and $\vec{c}$, the vectors $\overrightarrow{EC}$, $\overrightarrow{BF}$ and $\overrightarrow{FE}$.



3. Express the following vectors in terms of $\vec{p}$, $\vec{q}$, $\vec{r}$, $\vec{s}$ as denoted in the figure.



(i) $\overrightarrow{AC}$ (ii) $\overrightarrow{AD}$ (iii) $\overrightarrow{BE}$

If F is the midpoint of $\overrightarrow{AE}$

find:

(iv) $\overrightarrow{FB}$ (v) $\overrightarrow{FC}$ (vi) $\overrightarrow{FD}$

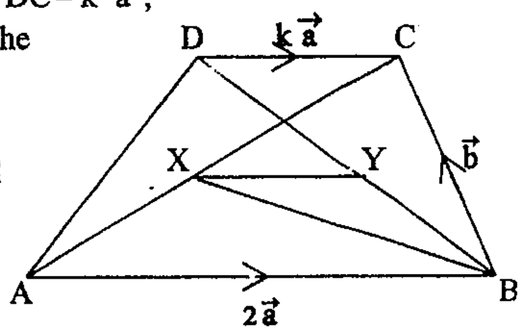
4. ABCDEF is a regular hexagon. $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

Express the followings in terms of $\vec{a}$ and $\vec{b}$.

(i) $\overrightarrow{CD}$ (ii) $\overrightarrow{EB}$ (iii) $\overrightarrow{FD}$ (iv) $\overrightarrow{AE}$ (v) $\overrightarrow{BF}$

5. ABCDEFGH is a regular octagon. If $\overrightarrow{AB}$ $\vec{a}$, $\overrightarrow{BC} = \vec{b}$, find in terms of $\vec{a}$, $\vec{b}$, the vectors $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}, \overrightarrow{FG}, \overrightarrow{GH}, \overrightarrow{HA}$.

6. In the figure, $\overrightarrow{AB} = 2\vec{a}$, $\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{DC} = k\,\vec{a}$, where $0 < k < 1$. Given that X and Y are the midpoints of AC and BD respectively,



express each of the vectors $\overrightarrow{AX}$, $\overrightarrow{BY}$ and $\overrightarrow{BX}$ in terms of $\vec{a}$ and $\vec{b}$.

Hence show that $\overrightarrow{XY} = (1 - \frac{k}{2})\,\vec{a}$.

7. If P is a point inside a parallelogram ABCD, prove that $\overrightarrow{PA} + \overrightarrow{PC} = \overrightarrow{PB} + \overrightarrow{PD}$.

8. In ΔABC, $\overrightarrow{BP} = \overrightarrow{PC}$ and $\overrightarrow{CQ} = \frac{1}{3}\overrightarrow{CA}$.

Prove that $2\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{BA} = 6\overrightarrow{PQ}$.

9. In the quadrilateral OABC, D is the midpoint of BC and G is a point on AD such that AG : GD = 2 : 1. If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$, express $\overrightarrow{OD}$ and $\overrightarrow{OG}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

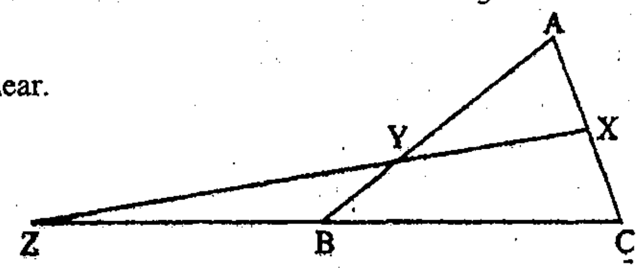
10. In a triangle ABC, $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$. If P, Q and R are the midpoints of BC, CA and AB respectively, express $\overrightarrow{BC}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ in terms of $\vec{a}$ and $\vec{b}$.

11. OPQR is a parallelogram and OR is produced to S such that OS = 3 OR. If Y is a point on OQ such that $\overrightarrow{OQ} = 4\overrightarrow{YQ}$, show that Y lies on PS.

12. In ΔABC, AB = BC. CB is produced to Z such that BC = BZ. X and Y are the points on AC and AB such that AX=XC and BY = $\frac{1}{3}$BA. Use a vector method to prove that

(i) X, Y, Z are collinear.

(ii) YZ = 2XY.



13. It is given that $\vec{a}$ and $\vec{b}$ are not parallel.

If $3\vec{a} + x(\vec{b} - \vec{a}) = y(\vec{a} + 2\vec{b})$, find the values of x and y. ($\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$)

14. Given that $\vec{p} = 2\vec{a} + 3\vec{b}$, $\vec{q} = 4\vec{a} - \vec{b}$ and $\vec{r} = h\,\vec{a} + (3h+k)\,\vec{b}$, where $\vec{a}$ and $\vec{b}$ are not parallel, calculate the value of h and of k when $2\vec{p} = 3\vec{q} - 4\vec{r}$. $(\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0})$

10.2 Applications to Elementary Geometry

So far we have developed the algebra of vectors using geometrical arguments; we now reverse the process and show that the algebra of vectors may usefully be employed to prove some results of Euclidean geometry by using geometric vectors.

Example 1.

Show that the diagonals of a parallelogram bisect each other.

Solution

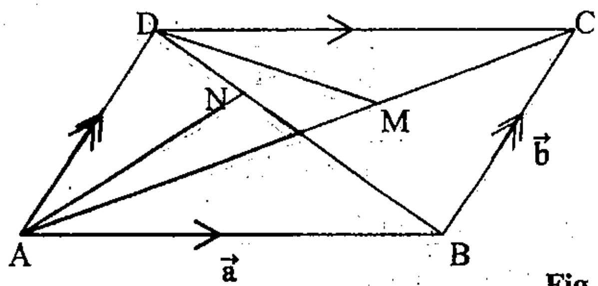


Fig. 10 . 14

In the parallelogram ABCD, let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$. Then $\overrightarrow{DC} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$. Assume that M is the midpoint of AC and N is the midpoint of BD.

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\vec{a} + \vec{b})$$

$$\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\vec{b} + \vec{a}$$

$$\overrightarrow{DN} = \frac{1}{2}\overrightarrow{DB} = \frac{1}{2}(\vec{a} - \vec{b})$$

$$\overrightarrow{AN} = \overrightarrow{AD} + \overrightarrow{DN} = \vec{b} + \frac{1}{2}(\vec{a} - \vec{b}) = \frac{1}{2}(\vec{a} + \vec{b})$$

Thus $\overrightarrow{AM} = \overrightarrow{AN}$.

This implies that M and N are the same point.

Example 2.

Given: Quadrilateral ABCD with P, Q, R, S the midpoints of the respective sides.

Prove: PQRS is a parallelogram.

Proof: By the polygon rule of vector addition, we have

$$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA} = \vec{0}$$

$$2\overrightarrow{PB}+2\overrightarrow{BQ}+2\overrightarrow{RD}+2\overrightarrow{DS} = \vec{0}$$

$$2(\overrightarrow{PB}+\overrightarrow{BQ})+2(\overrightarrow{RD}+\overrightarrow{DS}) = \vec{0}$$

$$\overrightarrow{PQ}+\overrightarrow{RS} = \vec{0}$$

$$\overrightarrow{PQ} = -\overrightarrow{RS}$$

$$\overrightarrow{PQ} = \overrightarrow{SR}$$

$\therefore$ PQ = SR and PQ // SR

Hence PQRS is a parallelogram.

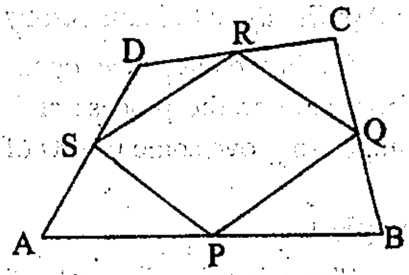
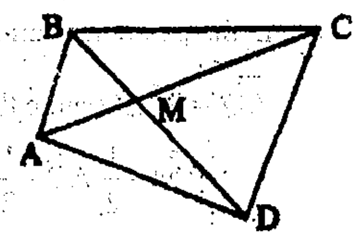


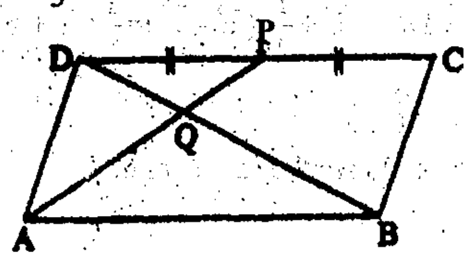
Fig.10.15

Exercise 10.2

1. In the figure MC = 2 MA and MD = 2 MB. Prove by a vector method that DC//AB and DC = 2 AB. (Hint: Let $\overrightarrow{AM}=\vec{a}$ and $\overrightarrow{MB}=\vec{b}$.)

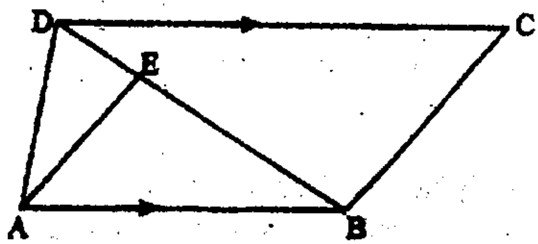


2. If P is the midpoint of the side CD of the parallelogram ABCD, prove by a vector method that DQ = $\frac{1}{3}$ DB.



3. Given: E is the point on the side BD of the ΔABD; $DE = \frac{1}{4}DB$, $AB // DC$ and $DC = \frac{4}{3}AB$. $\overrightarrow{AB} = 12\vec{a}$, $\overrightarrow{AD} = 4\vec{b}$.

Prove: $BC // AE$.



4. In ΔABC, D is the midpoint of BC, and AD is produced to E such that $\overrightarrow{AE} = 2\overrightarrow{AD}$. Prove by a vector method that CE is congruent and parallel to AB.

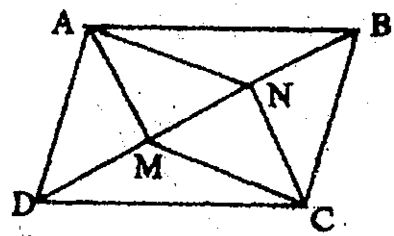
5. By using geometric vectors, show that the line segment joining the midpoints of two sides of any triangle is equal in length to half and parallel to the third side.

6. Prove, by a vector method, that if the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.

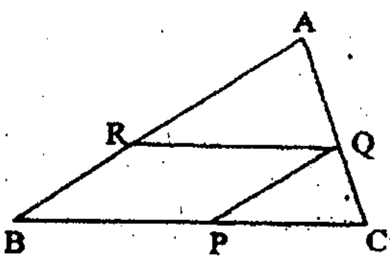
7. In the figure **ABCD is a parallelogram** and DM = MN = NB.

Prove by a **vector method that** ANCM is a parallelogram.

(Hint: Let $\overrightarrow{DA} = \vec{a}$ and $\overrightarrow{MD} = \vec{b}$.)



8. In the figure, P,Q and R are points on the sides of ΔABC such that BP=2PC, QA=2CQ and AR = 2 RB. Prove by a vector method that PQRB is a parallelogram. (Hint: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ and $\overrightarrow{AB} = 3\overrightarrow{RB}$,...)



9. The median AD of Δ ABC is produced to K so that $\overrightarrow{DK} = \frac{1}{3}\overrightarrow{AD}$.

If $\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD}$, prove that BKCG is a parallelogram.

10.3 Position Vectors

The position of the point A relative to the origin O is denoted by $\overrightarrow{OA}$. This vector $\overrightarrow{OA}$ is called **position vector** of A. $\overrightarrow{OA}$ can also be represented by $\vec{a}$.

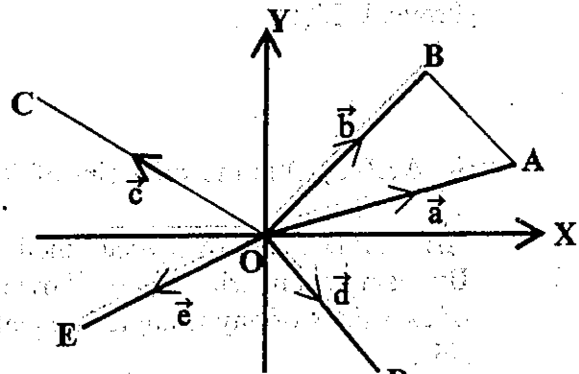


Fig. 10.16

Thus, given the origin O, we may refer to all points in the plane by their position vectors relative to this origin O.

To every geometric vector there corresponds a position vector of the same magnitude and direction.

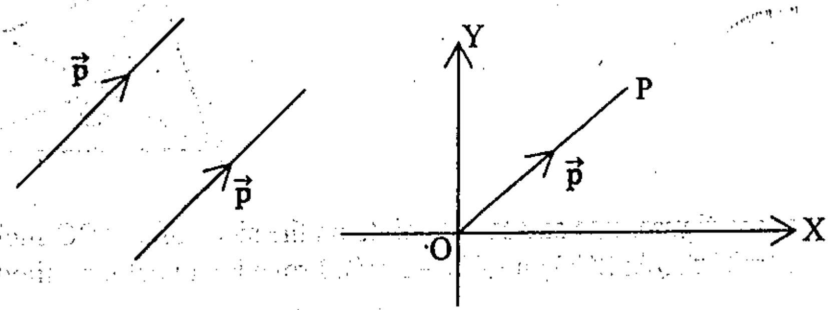


Fig. 10.17

A vector between two points A and B can be expressed in terms of their position vectors. From Fig. 10.16, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \vec{b} - \vec{a}$$

Similarly, $\overrightarrow{BA} = \vec{a} - \vec{b}$

Example 1.

The position vectors of three points P. Q and R, relative to an origin O, are $9\vec{a} - 4\vec{b}$, $-3\vec{a} - \vec{b}$ and $5\vec{a} - 3\vec{b}$ respectively. Express $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ in terms of $\vec{a}$ and $\vec{b}$. Are P, Q and R collinear?

Solution

$$\overrightarrow{OP} = 9\vec{a} - 4\vec{b}, \quad \overrightarrow{OQ} = -3\vec{a} - \vec{b}, \quad \overrightarrow{OR} = 5\vec{a} - 3\vec{b}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (-3\vec{a} - \vec{b}) - (9\vec{a} - 4\vec{b})$$

$$= -12\vec{a} + 3\vec{b} = -3(4\vec{a} - \vec{b})$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$$

$$= (5\vec{a} - 3\vec{b}) - (-3\vec{a} - \vec{b}) = 8\vec{a} - 2\vec{b} = 2(4\vec{a} - \vec{b})$$

$$2\overrightarrow{PQ} = -3\overrightarrow{QR}$$

$$\therefore \overrightarrow{PQ} = -\frac{3}{2}\overrightarrow{QR}$$

Thus P, Q and R are collinear.

Example 2.

Given that $\overrightarrow{OP} = 2\vec{a} + \vec{b}$, $\overrightarrow{OQ} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OR} = h\vec{a} + 5\vec{b}$, find, in terms of $\vec{a}$ and $\vec{b}$, the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. If P, Q and R are collinear, find the value of h.

Solution

$$\overrightarrow{OP} = 2\vec{a} + \vec{b}, \qquad \overrightarrow{OQ} = 3\vec{a} - 2\vec{b}, \qquad \overrightarrow{OR} = h\vec{a} + 5\vec{b}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (3\vec{a} - 2\vec{b}) - (2\vec{a} + \vec{b}) = \vec{a} - 3\vec{b}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (h\vec{a} + 5\vec{b}) - (2\vec{a} + \vec{b})$$

$$= (h - 2)\vec{a} + 4\vec{b}$$

If P, Q and R are collinear,

$\overrightarrow{PQ} = k\,\overrightarrow{PR}$ where k is a constant.

$$\vec{a} - 3\vec{b} = k[(h-2)\vec{a} + 4\vec{b}] = k(h-2)\vec{a} + 4k\vec{b}$$

By Corollary 1.1, we have

$k(h - 2) = 1$ and $4k = -3$.

Solving these equations, we get

$$h = \frac{2}{3}.$$

Now, we will establish a result known as the Section Formula. This formula enables us to write down the position vector of a point on a given line segment.

Theorem 2 (The Section Formula)

If APB is a line segment, with AP : PB = m : n, and if the position vectors of **A,P,B** relative to an origin O, are $\vec{a}$, $\vec{p}$, $\vec{b}$ respectively, then

$$\vec{p} = \frac{1}{m+n}(m\,\vec{b} + n\,\vec{a}\,)$$

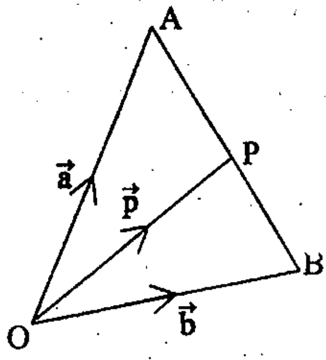


Fig. 10.18

Proof

We have , $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{p} - \vec{a}$$

From $\dfrac{AP}{PB} = \dfrac{m}{n}$, we have

$$\frac{AP}{AP+PB} = \frac{m}{m+n}$$

$$\frac{AP}{AB} = \frac{m}{m+n}$$

Thus $\overrightarrow{AP} = \dfrac{m}{m+n}\overrightarrow{AB}$, and so

$$\vec{p} - \vec{a} = \frac{m}{m+n}(\vec{b} - \vec{a}).$$

If follows that

$$\vec{p} = \frac{1}{m+n}(m\,\vec{b} + n\,\vec{a}\,).$$

Corollary 2.1 (The Midpoint Formula)

If P is the midpoint of AB, then $\vec{p} = \frac{1}{2}(\vec{a} + \vec{b})$,

where $\vec{p}$, $\vec{a}$, $\vec{b}$ are the position vectors of P, A, B, relative to an origin O. respectively.

Proof

Since AP = PB, we have

AP:PB = 1 :1 = m:n.

Thus

$$\vec{p} = \frac{1}{1+1}(1\vec{b} + 1\vec{a}) = \frac{1}{2}(\vec{a} + \vec{b}).$$

Example 3.

If $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, find the position vector of R in terms of $\vec{a}$ and $\vec{b}$ if R divides AB in the ratio (i) 1:2 (ii) –5:2.

Solution

Let $\vec{r}$ be the position vector of R relative to O.

(i) $\vec{r} = \frac{1}{1+2}(1\vec{b} + 2\vec{a}) = \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b}$

(ii) $\vec{r} = \frac{1}{-5+2}(-5\vec{b} + 2\vec{a}) = -\frac{2}{3}\vec{a} + \frac{5}{3}\vec{b}$

Example 4.

The position vectors of three points A, B and C, relative to an origin O, are $3\vec{p} + 2\vec{q}$, $-5\vec{p} - 3\vec{q}$ and $4\vec{p} - \vec{q}$ respectively. The midpoint of AB is M and the point N is such that $\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC}$. Find $\overrightarrow{MN}$ in terms of $\vec{p}$ and $\vec{q}$.

Solution

$$\overrightarrow{OA} = 3\vec{p} + 2\vec{q},\ \overrightarrow{OB} = -5\vec{p} - 3\vec{q},\ \overrightarrow{OC} = 4\vec{p} - \vec{q}$$

M is the midpoint of AB.

$$\therefore \overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$$

$$= \frac{1}{2}(3\vec{p}+2\vec{q}-5\vec{p}-3\vec{q}) = -\vec{p}-\frac{1}{2}\vec{q}$$

$$\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\overrightarrow{OC}-\overrightarrow{OA})$$

$$= \frac{1}{3}(4\vec{p}-\vec{q}-3\vec{p}-2\vec{q}) = \frac{1}{3}\vec{p}-\vec{q}$$

$$\overrightarrow{ON} = \overrightarrow{OA}+\overrightarrow{AN}$$

$$= 3\vec{p}+2\vec{q}+\frac{1}{3}\vec{p}-\vec{q} = \frac{10}{3}\vec{p}+\vec{q}$$

Thus $\overrightarrow{MN} = \overrightarrow{ON}-\overrightarrow{OM} = (\frac{10}{3}\vec{p}+\vec{q})-(-\vec{p}-\frac{1}{2}\vec{q}) = \frac{13}{3}\vec{p}+\frac{3}{2}\vec{q}$.

Example 5.

Prove by vectors that the medians of a triangle are concurrent.

Proof

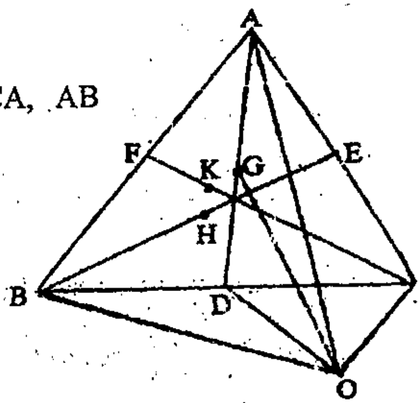


Fig.10.19

Let D, E, F be the midpoints of BC, CA, AB respectively.
Referring to an origin O, let

$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$.

Then

$$\overrightarrow{OD} = \frac{1}{2}(\vec{b}+\vec{c}).$$

If G is a point on AD such that AG : GD = 2 : 1 , then

$$\overrightarrow{OG} = \frac{1}{2+1}(2\overrightarrow{OD}+1\overrightarrow{OA})$$

$$= \frac{1}{3}\{2.\ \frac{1}{2}(\vec{b}+\vec{c})+\vec{a}\} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c}).$$

Similarly, if H and K are points on BE and CF , respectively, such that BH : HE = 2:1 = CK: KF, we can show that

$$\overrightarrow{OH} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c}) \quad \text{and} \quad \overrightarrow{OK} = \frac{1}{3}(\vec{a}+\vec{b}+\vec{c}).$$

$$\therefore \quad \overrightarrow{OG} = \overrightarrow{OH} = \overrightarrow{OK}$$

It means that G, H and K are identical.

Hence the three medians meet at one point.

Note:

The point where the medians meet is called the centroid of the given triangle.

Exercise 10.3

1. The position vectors of P,Q and R with respect to an origin O are $3\vec{b}+5\vec{c}-2\vec{a}$, $\vec{a}+2\vec{b}+3\vec{c}$ and $7\vec{a}-\vec{c}$. Are the points P, Q and R collinear?

2. The position vectors of A, B and C are $2\vec{p}-\vec{q}$, $k\vec{p}+\vec{q}$ and $12\vec{p}+4\vec{q}$ respectively. Calculate the value of k if A,B and C are collinear.

3. The position vectors of A, B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$ for each of the following cases.

   (i) $3\overrightarrow{AC} = \overrightarrow{CB}$ (ii) $3\overrightarrow{BC} = 5\overrightarrow{CA}$ (iii) $\overrightarrow{AC} = -2\overrightarrow{CB}$ (iv) $\overrightarrow{AB} = 4\overrightarrow{BC}$

4. The position vectors of points P and Q relative to an **origin O** are $2\vec{a}+5\vec{b}$ and $3\vec{a}-7\vec{b}$ respectively. R is a point on PQ such that PR: RQ =1:3. Find the position vector of R, relative to O, in terms of $\vec{a}$ and $\vec{b}$.

5. The position vectors of A, B, P are $\vec{a}$, $\vec{b}$, $\vec{p}$. In those cases where A, B, P are collinear, calculate the ratio AP : PB if $\vec{p}$ has the following values.

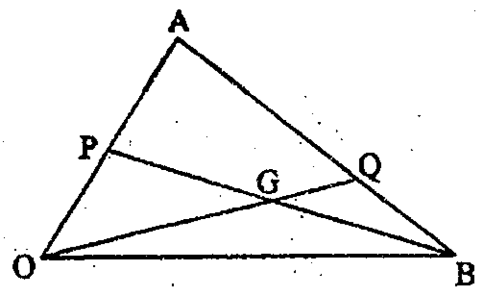
(i) $\frac{1}{3}(\vec{a}+2\vec{b})$ (ii) $\frac{1}{2}(3\vec{a}-\vec{b})$ (iii) $\frac{7}{5}\vec{a}-\frac{2}{5}\vec{b}$

6. In the quadrilateral ABCD, M and N are the midpoints of AC and BD respectively. Prove that $\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{AD} + \overrightarrow{CD} = 4\overrightarrow{MN}$.

7. In the diagram, P is the midpoint of OA and Q lies on AB such that $\overrightarrow{AQ} = 3\overrightarrow{QB}$. Given that $\overrightarrow{OA} = 5\vec{s}$ and $\overrightarrow{OB} = 10\vec{t}$, express in terms of $\vec{s}$ and $\vec{t}$,

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{BQ}$ (iii) $\overrightarrow{OQ}$ (iv) $\overrightarrow{BP}$.

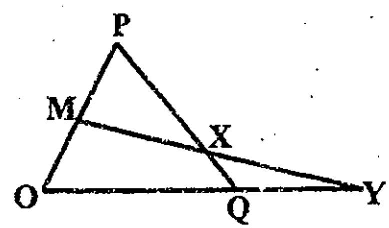
Given that $\overrightarrow{BG} = \lambda\,\overrightarrow{BP}$ and $\overrightarrow{OG} = \mu\,\overrightarrow{OQ}$, evaluate λ and μ.



8. In the diagram, M is the midpoint of OP, and, Q, the midpoint of OY. $\overrightarrow{OP} = 2\vec{a}$ and $\overrightarrow{OQ} = 6\vec{b}$. Express in terms of $\vec{a}$ and $\vec{b}$,

(i) $\overrightarrow{PQ}$ (ii) $\overrightarrow{OY}$ (iii) $\overrightarrow{MY}$.

If $\overrightarrow{XQ} = \lambda\,\overrightarrow{PQ}$ and $\overrightarrow{XY} = \mu\,\overrightarrow{MY}$, evaluate λ and μ.



9. Points A and B have position vectors $\vec{a}$ and $\vec{b}$ respectively, relative to an origin O. The point C lies on OA produced such that OC = 3OA, and D lies on OB such that $OD = \frac{1}{4}$ OB. Express $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in terms of $\vec{a}$ and $\vec{b}$. The line segments AB and CD intersect at P. If CP = h CD and AP = k AB, calculate the values of h and k.

10 If G is the centroid of a triangle ABC, show that

(i) $\overrightarrow{AB}+\overrightarrow{AC}=3\,\overrightarrow{AG}$ (ii) $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{0}$.

10.4 Two-Dimensional Vectors

By taking coordinate axes, we can express any given position vector by the coordinates (x, y) of its terminal point. A column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ is used to denote the corresponding position vector. Thus, if A is the point (x, y),

$$\overrightarrow{OA} = \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

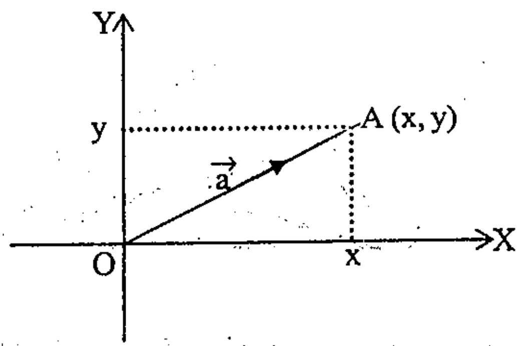


Fig. 10.20

Definition 8

A unit vector is a vector whose magnitude is 1.

The unit vector in the direction of $\vec{a}$, denoted by $\hat{a}$, is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$

In Fig. 10.21, the coordinates of I and J are (1,0) and (0,1) respectively. Thus

$$\overrightarrow{OI} = \hat{i} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \overrightarrow{OJ} = \hat{j} = \begin{pmatrix}0\\1\end{pmatrix}.$$

We note that $\hat{i}$ and $\hat{j}$ are of unit length and parallel to the X-axis and Y-axis respectively. The vectors $\hat{i}$ and $\hat{j}$ are called unit vectors in the positive direction of the X-axis and Y-axis respectively.

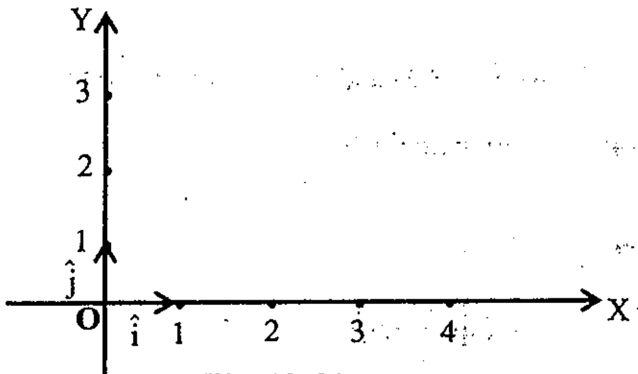


Fig. 10. 21

Any vectors in the plane can be expressed in cerms of $\hat{i}$ nd $\hat{j}$. If A is the point (x, y), we have

$$\vec{a} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix} + \begin{pmatrix}0\\y\end{pmatrix}$$

$$= x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$$

$$= x\hat{i} + y\hat{j}.$$

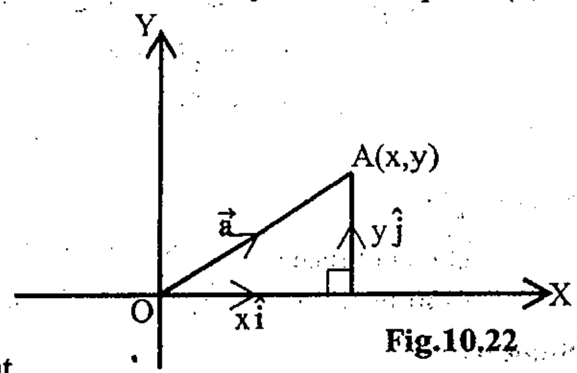


Fig.10.22

From Fig. 10.22, it can be seen that

$$\left|\overrightarrow{OA}\right| = |\vec{a}| = \left|\begin{pmatrix}x\\y\end{pmatrix}\right| = |x\hat{i} + y\hat{j}| = \sqrt{x^2 + y^2}$$

Hence

$$\hat{a} = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2}} \end{pmatrix} = \frac{x}{\sqrt{x^2+y^2}}\,\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{j}.$$

Example 1.

If P = (3,4), R= (8,2) and O is the origin and $\overrightarrow{OT} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OR}$, find the coordinates of the point T.

Solution

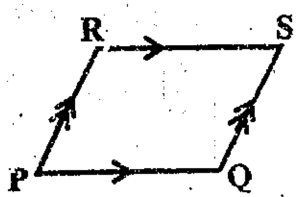
$$\begin{aligned}
\text{Let } T &= (x, y). \\
\overrightarrow{OT} &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OR} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\
\therefore x &= 7 \text{ and } y = 5 \\
\therefore T &= (7, 5)
\end{aligned}$$

Example 2.

The coordinates of P, Q and R are (1,2) , (7,3) and (4,7) respectively. Find the coordinates of S if PQSR is a parallelogram.

Solution

Let the coordinates of S be (h, k).



$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \overrightarrow{OQ} = \begin{pmatrix} 7 \\ 3 \end{pmatrix},\ \overrightarrow{OR} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \text{and } \overrightarrow{OS} = \begin{pmatrix} h \\ k \end{pmatrix}$$

Since PQSR is a parallelogram, $\overrightarrow{PQ} = \overrightarrow{RS}$.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \begin{pmatrix} h \\ k \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} h-4 \\ k-7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} h-4 \\ k-7 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$

$\therefore$ $h - 4 = 6$ and $k - 7 = 1$

i.e. $h = 10$ and $k = 8$

The coordinates of S are (10, 8).

Note:

Previously we have met the two equivalent definitions of addition of two vectors, namely the triangle rule and the parallelogram rule.

The triangle rule : $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

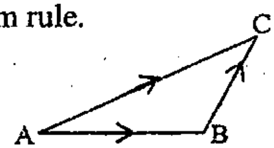
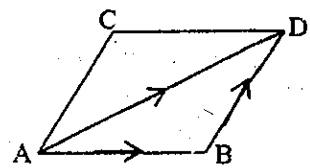


Fig. 10.23



The parallelogram rule : $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$

Fig. 10.24

Now in the case of two-dimensional vectors, if A = (x, y) then the position vector of A relative to origin O, i.e. $\overrightarrow{OA}$ is defined by the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$. We have freely used matrix addition to add two vectors. We shall show that this method gives the same value for the sum of two vectors as that given by the triangle rule or the parallelogram rule.

Let A(x, y), B(u, v) be given.

Then by definition $\overrightarrow{OA} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} u \\ v \end{pmatrix}$.

Complete the parallelogram OACB.

Then by parallelogram rule $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$.

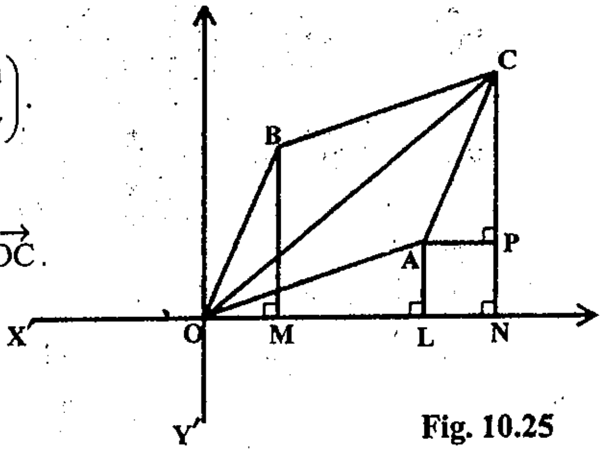


Fig. 10.25

Let the perpendiculars be drawn as shown in the figure. Since OACB is a parallelogram it is easy to see that $\Delta OBM \cong \Delta ACP$.

Therefore LN = AP = OM = u and CP = BM = v. Hence ON = OL + LN = x + u, CN = CP + PN = CP + AL = v + y. Thus $\overrightarrow{OC} = \begin{pmatrix} x+u \\ v+y \end{pmatrix}$ which is precisely the same as the sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ by matrix method.

Similarly in the case of multiplication of a vector by a scalar(i.e. a real number) it can be shown that we get the same value whether we use the previous definition 7 or multiplication of a matrix by a real number.

Example 3.

If $\vec{a} = 5\hat{i} + 4\hat{j}$ and $\vec{b} = 2\hat{i} - \hat{j}$, find the following vectors in terms of $\hat{i}$ and $\hat{j}$, and as column vectors.

(i) $3\vec{a} + 2\vec{b}$ (ii) $2\vec{a} - \vec{b}$ (iii) $-\vec{a} + 4\vec{b}$

Solution

(i) $3\vec{a} + 2\vec{b} = 3(5\hat{i} + 4\hat{j}) + 2(2\hat{i} - \hat{j})$

$= 15\hat{i} + 12\hat{j} + 4\hat{i} - 2\hat{j}$

$= 19\hat{i} + 10\hat{j} = \begin{pmatrix} 19 \\ 10 \end{pmatrix}$

(ii) $2\vec{a} - \vec{b} = 2(5\hat{i} + 4\hat{j}) - (2\hat{i} - \hat{j})$

$= 10\hat{i} + 8\hat{j} - 2\hat{i} + \hat{j} = 8\hat{i} + 9\hat{j} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$

(iii) $-\vec{a} + 4\vec{b} = -(5\hat{i} + 4\hat{j}) + 4(2\hat{i} - \hat{j})$

$= -5\hat{i} - 4\hat{j} + 8\hat{i} - 4\hat{j} = 3\hat{i} - 8\hat{j} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$

Example 4.

The vector $\overrightarrow{OA}$ **has the magnitude of** 39 units and has the same direction as $5\hat{i} + 12\hat{j}$. The vector $\overrightarrow{OB}$ has the magnitude of 25 units and has the same direction as $-3\hat{i} + 4\hat{j}$. Express $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in terms of $\hat{i}$ and $\hat{j}$ and find the magnitude of $\overrightarrow{AB}$.

Solution

Let $\vec{p} = 5\hat{i} + 12\hat{j}$ and $\vec{q} = -3\hat{i} + 4\hat{j}$

$|\vec{p}| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

$\hat{p} = \dfrac{\vec{p}}{|\vec{p}|} = \dfrac{1}{13}(5\hat{i} + 12\hat{j})$

$$\overrightarrow{OA} = 39\hat{p} = 39 \times \frac{1}{13}(5\hat{i} + 12\hat{j}) = 15\hat{i} + 36\hat{j}$$

$$|\overrightarrow{q}| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

$$\hat{q} = \frac{\overrightarrow{q}}{|\overrightarrow{q}|} = \frac{1}{5}(-3\hat{i} + 4\hat{j})$$

$$\therefore \overrightarrow{OB} = 25\hat{q} = 25 \times \frac{1}{5}(-3\hat{i} + 4\hat{j}) = -15\hat{i} + 20\hat{j}$$

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (-15\hat{i} + 20\hat{j}) - (15\hat{i} + 36\hat{j}) \\ &= -30\hat{i} - 16\hat{j} \end{aligned}$$

$$|\overrightarrow{AB}| = \sqrt{(-30)^2 + (-16)^2} = \sqrt{1156} = 34$$

Exercise 10.4

1. The position vectors of A, B and C are $\overrightarrow{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\overrightarrow{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{c} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ respectively. Calculate the modulus of the vector

   (i) $\overrightarrow{a} + 2\overrightarrow{b} + 2\overrightarrow{c}$ (ii) $3\overrightarrow{a} - 3\overrightarrow{b} + 4\overrightarrow{c}$ (iii) $8\overrightarrow{a} + 6\overrightarrow{b} - \overrightarrow{c}$.

2. The coordinates of P, Q and R are (1,0), (4,2) and (5,4) respectively. Use vector method to determine the coordinates of S if

   (i) PQRS is a parallelogram,

   (ii) PRQS is a parallelogram.

3. The coordinates of A, B and C are (1,2), (7,1) and (– 3,7) respectively. If O is the origin and $\overrightarrow{OC} = h\,\overrightarrow{OA} + k\overrightarrow{OB}$, where h and k are constants, find the values of h and k.

4. A,B and C are points with position vectors $\hat{i}+3\hat{j}, 2\hat{i}+5\hat{j}$ and $k\,\hat{i} - 4\,\hat{j}$ respectively. Find the value of k if (i) A, B and C are collinear, (ii) $|\overrightarrow{AC}| = 4\,|\overrightarrow{AB}|$

5. The position vectors of A and B relative to an origin O are $\begin{pmatrix} 4 \\ 14 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 2 \end{pmatrix}$ respectively. Given that C lies on AB and has position vector $\begin{pmatrix} 2t \\ t \end{pmatrix}$, find the value of t and the ratio AC : CB .

6. Points P and Q have position vectors $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ respectively, relative to an origin O. Given that point R with position vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$ lies on PQ produced, calculate (i) the value of k, (ii) the value of $|2\,\overrightarrow{PQ}+\overrightarrow{OR}\,|$.

7. The vector $\overrightarrow{OP}$ has a magnitude of 26 units and has the same direction as $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$. The vector $\overrightarrow{OQ}$ has a magnitude of 20 units and has the same direction as $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as column vectors and find the unit vector in the direction of $\overrightarrow{PQ}$.

8. The three points O, P and Q are such that $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} q \\ 2q \end{pmatrix}$. Given that $\overrightarrow{PQ}$ is a unit vector, calculate the possible values of q .

9. The position vectors, relative to an origin O, of the points L and M are $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ respectively. Given that $\overrightarrow{ON}$ is the unit vector parallel to $\overrightarrow{LM}$, find the position vector of N relative to O.

10.5 Transformation Geometry

Geometry is the study of shape, position and size. A figure which is moved to a new position, or has its shape or size altered is said to undergo a transformation. Matrices are able to describe various transformations.

When a plane figure is moved to a new position without alteration of its shape or size, distances between points and angles between lines within the figure are preserved by this transformation. A transformation in which shape is retained and size altered preserves angles between lines within the figure but not distances between points. When size is retained, the transformation preserves the area of the shape.

In a certain transformation a point P on the original figure becomes some point P' on the transformed figure. We say that P is mapped into P' by the transformation. We can consider the geometrical transformations as taking place in the coordinate plane.

Let a point P (x , y) be mapped into a point P' (x', y') by a transformation. Then we can get a relation between coordinates as of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a matrix. This matrix A is called the transformation matrix.

10.6 Transformations which Preserve Distances and Angles .

The Reflection Matrix

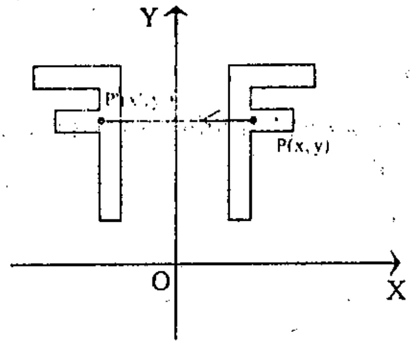


Fig. 10.26

When the reflection takes place in the line OY, any point P (x,y) is mapped into a point P' = (x', y') where

$$x' = -x$$
$$y' = y$$

since , by the reflection, the y coordinate is unchanged, but the x coordinate has the sign changed. We may write the above equations in the form

$$x' = -1x + 0y$$
$$y' = 0x + 1y$$

and now put these equations into matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ has the geometrical property of reflection in the line OY.

Example 1.

Find the map of the point (–2,3) by the matrix F.

Solution

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\therefore x' = 2, \; y' = 3$

The mapped point is $(x', y') = (2, 3)$.

The Rotation Matrix

The matrix of rotation about the origin O (anticlockwise) through an angle θ is given by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Proof

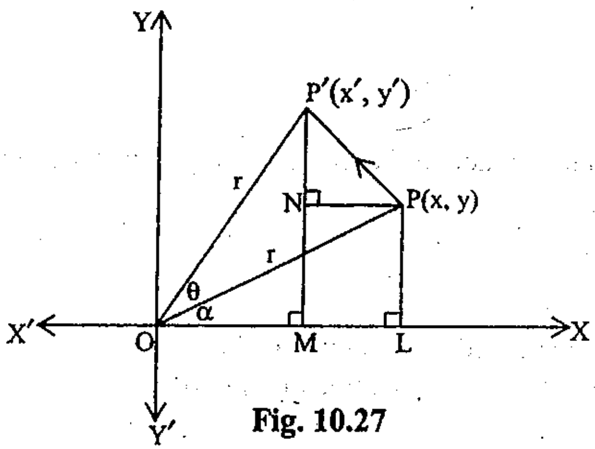


Fig. 10.27

Suppose P(x, y) is any given point with $\angle POX = \alpha$. Under the transformation considered; let P′ be the image of P so that $\angle POP' = \theta$. Let the perpendiculars be drawn as shown in the figure.

Let $OP = OP' = r$.

Now $x' = OM = r\cos(\alpha + \theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$= OL\cos\theta - PL\sin\theta = x\cos\theta - y\sin\theta$.

Also $y' = P'M = r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = y\cos\theta + x\sin\theta$.

$$\text{Thus } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore the required matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Note:

We have used the compound angle formulae $\cos(\alpha + \theta) = \cos\alpha \cos\theta - \sin\alpha \sin\theta$, $\sin(\alpha + \theta) = \sin\alpha \cos\theta + \cos\alpha \sin\theta$. These formulae can be derived without use of idea of rotation. (See 11.7 of Chapter 11)

Example 2.

Find the matrix which rotates through 45° and find the map of the point (1,1).

Solution

$$R = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\text{Let } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

$$\therefore x' = 0, \quad y' = \sqrt{2}$$

The mapped point is $(x', y') = (0, \sqrt{2})$.

The Translation Matrix

A point P(x,y) is translated horizontally through a distance h and vertically through a distance k to the point P' (x', y').

Then $x' = x + h$,

$y' = y + k$,

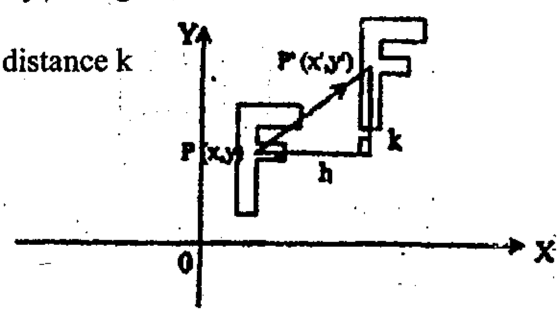


Fig.10.28

In matrix form, these equations are

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

This time a 2×3 matrix is required as we have values h and k unattached to x and y. It is more convenient to have a square matrix, so the 2×3 matrix is made into a 3×3 as $\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ and the new matrix form is

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The matrix $L = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ describes a translation which maps the origin to the point (h, k).

Example 3.

Find the matrix which will translate a distance of –2 units horizontally and 2 units vertically. What is the map of $(1, -4)$?

Solution

We have $h = -2$, $k = 2$. So

$$L = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Then $$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

$$\therefore x' = -1, \quad y' = -2$$

The mapped point is $(x', y') = (-1, -2)$.

We have found matrices to describe certain transformations. As we can combine transformations by performing one after the other, we can use our matrices together to describe successive transformations. By using matrix multiplication we can find any combination of geometrical transformations.

Note that the order of matrix multiplication comes out backwards. If we had a transformation described by a matrix A followed by a transformation described by a matrix B, we need to compute BA. We can explain it. Let a point P (x,y) be mapped into the point P' (x',y') by the transformation matrix A and P' (x',y') be mapped again into the point P" (x'',y'') by the transformation matrix B.

At first, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

and then

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = B \begin{pmatrix} x' \\ y' \end{pmatrix} = BA \begin{pmatrix} x \\ y \end{pmatrix}$$

This gives the combined transformation matrix as BA.

Example 4.

Find the matrix which will rotate 30° and then reflect in the line OY. What is the map of the point (1,0)?

Solution

$$R = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The transformation matrix is

$$T = FR = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$.

$$\therefore x' = -\frac{\sqrt{3}}{2}, \quad y' = \frac{1}{2}$$

The mapped point is (x', y')= $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

To use the translation 3×3 matrix with another 2×2 matrix we shall need to enlarge the 2×2 matrix into a 3×3. The 2×2 reflection matrix

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ becomes $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by placing 0 0 1 as the last row and column.

Similarly, the 3×3 rotation matrix is $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Exercise 10.5

1. Find the maps of the points $(-1,-2)$, $(4, 3)$ and $(3,-4)$ by the matrix F.
2. Show that the matrix S which reflects in the line OX is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and find the maps of the points $(-2,-1)$, $(3, 4)$ and $(-4, 3)$ by S.
3. Find the matrices which rotate through 60°, 90° and 180° and find the respective maps of the points $(0,2)$, $(-1,1)$ and $(2,-1)$.
4. Find the matrix Z which will rotate through an angle θ clockwise about O.
5. Find the matrix which will reflect in the line OY followed by a rotation through 60°. What is the map of the point $(-1, 0)$?
6. Find the matrix which will translate through 3 units horizontally and 1 unit vertically followed by a rotation through 45°, and find the map of the point $(1, 2)$.
7. Find the matrix which will reflect in the line OY followed by a translation through 2 units horizontally and -1 unit vertically. What is the map of the point $(1, 2)$?
8. Find the matrix which will reflect in the line $y = x$. Find the map of the point $(1,2)$ when it is reflected in the line $y = x$.
   [Consider using matrices which rotate through an angle 45° clockwise about O followed by a reflection in the line OX followed by a rotation through 45°.]

SUMMARY

1. **Important Words and Symbols**

   Vectors and Scalars

   Geometric vectors $\overrightarrow{AB}$

   Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = AB$

   Negative of a vector $\vec{a} = -\vec{a}$

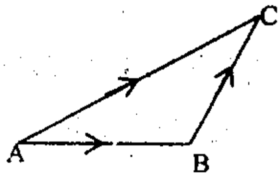
   Zero vector $= \vec{0}$

   Position vector

2. **Important definitions and formulae**

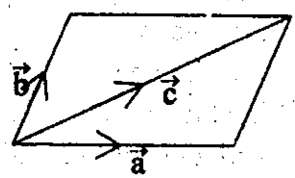
(i) $\vec{a} = \vec{b} \Rightarrow \vec{a} \parallel \vec{b}$ and $|\vec{a}| = |\vec{b}|$

(ii) The Triangle Rule of Vector Addition



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(iii) The Parallelogram Rule of Vector Addition



$\vec{a} + \vec{b} = \vec{c}$

(iv) The Polygon Rule of Vector Addition

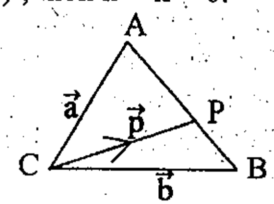
If $A_1 A_2 A_3 \ldots A_n$ is a closed polygon, then

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots + \overrightarrow{A_{n-1}A_n} = \overrightarrow{A_1A_n}.$$

(v) If $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$, then $(\vec{a} = k\vec{b} \quad (k \neq 0) \Leftrightarrow \vec{a} \parallel \vec{b})$

(vi) If $h\vec{a} = k\vec{b} \quad (\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}, \vec{a} \nparallel \vec{b})$, then $h = k = 0$.

(vii) If $\dfrac{AP}{PB} = \dfrac{m}{n}$, then $\vec{p} = \dfrac{m\vec{a} + n\vec{b}}{m+n}$



3. **Transformation Matrices**

(i) Reflection matrices: $F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(ii) Rotation matrix: $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(iii) Translation matrix: $L = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

CHAPTER 11

Trigonometry

Nowadays it is known that there is more to learn in trigonometry than measuring triangles. In fact, trigonometry generally deals with angles of all sizes with degree measure not necessarily confined to angle of triangles. In this section we introduce negative angles.

11.1 Trigonometric Ratios for Special Angles

We have studied the trigonometric ratios for the special angles measured in degrees. Table shows these ratios for the angle measured either in radians or degrees.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
$\frac{\pi}{6} = (30°)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4} = (45°)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3} = (60°)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

Table 11.1

11.2 Trigonometric Ratio of Any Angle

Consider a circle of unit radius with its centre at the origin of the XY-plane.

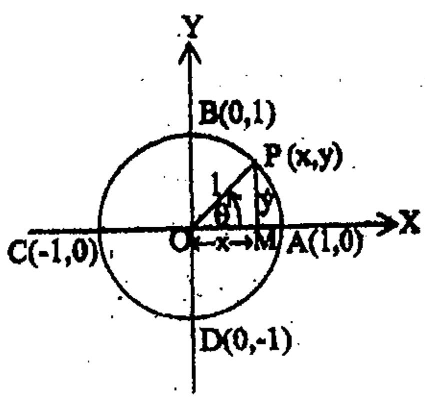


Fig. 11.1

Suppose a point P starts at A (1, 0) and moves $|\theta|$ units around the circumference of the circle, counter clockwise if $\theta > 0°$ and clockwise if $\theta < 0°$.

We can locate the exact position of P for any specific value of θ. If $\theta > 2\pi$ we shall continue around the circle post A until we have covered the entire circle. Now to every value of θ there is associated a point called the terminal point, whose distance along the arc from A (1, 0) is $|\theta|$ units. If we designate this terminal point by P(x,y). We define the trigonometric ratios for θ as follows:

$$\cos\theta = x \qquad \sec\theta = \frac{1}{x}$$

$$\sin\theta = y \qquad \text{cosec}\,\theta = \frac{1}{y}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

It follows that

From,

$$x^2 + y^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \text{cosec}^2\,\theta$$

$$\sec\theta = \frac{1}{\cos\theta},\quad \cos\theta \neq 0 \qquad \text{cosec}\,\theta = \frac{1}{\sin\theta},\quad \sin\theta \neq 0$$

$$\cot\theta = \frac{1}{\tan\theta},\quad \tan\theta \neq 0 \qquad \tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cos\theta \neq 0$$

Looking at the unit circle of Fig. 11.1, it is clear that $-1 \leq x \leq 1$, and $-1 \leq y \leq 1$.

i.e. $-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$.

The signs of $x = \cos\theta$ and $y = \sin\theta$ depend upon the quadrant in which P(x,y) lies.

(S) II	I (A)
$\sin\theta$ (+) cosec θ(+) others (−)	All (+)
(T) III	IV (C)
$\tan\theta$ (+) $\cot\theta$ (+) others (−)	$\cos\theta$ (+) $\sec\theta$ (+) others (−)

Fig. 11.2

11.3 Negative Angles

Consider a **unit circle with a point** P(x,y) and OP rotates through an angle θ from the X-axis.

Under a reflection in the X-axis, the point P is mapped onto the point Q(x', y').

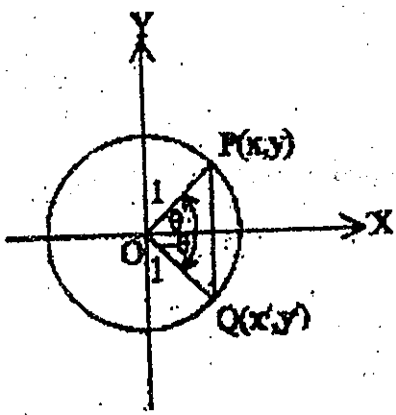


Fig. 11.3

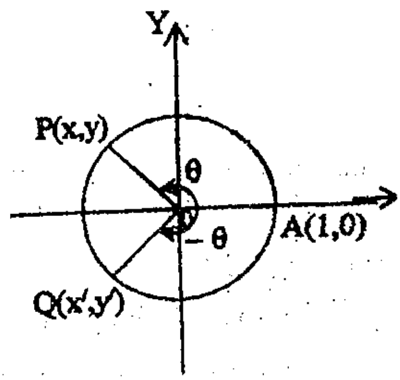


Fig. 11.4

From the figure, $\cos(-\theta) = x' = x = \cos\theta$

$$\sin(-\theta) = y' = -y = -\sin\theta$$

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{y'}{x'} = \frac{-y}{x} = -\tan\theta$$

11.4 Basic Identities

Using a right triangle ABC, we have

$$\sin(90^\circ - \theta) = \cos\theta$$

$$\cos(90^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ - \theta) = \cot\theta$$

$$\cot(90^\circ - \theta) = \tan\theta$$

$$\sec(90^\circ - \theta) = \mathrm{cosec}\,\theta$$

$$\mathrm{cosec}(90^\circ - \theta) = \sec\theta$$

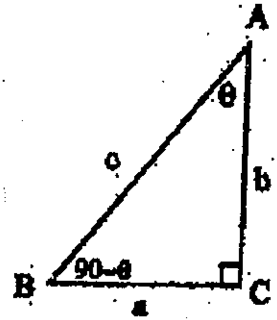


Fig. 11.5

Consider a unit circle with a point P (x,y) and OP rotating through an angle θ from the X-axis.

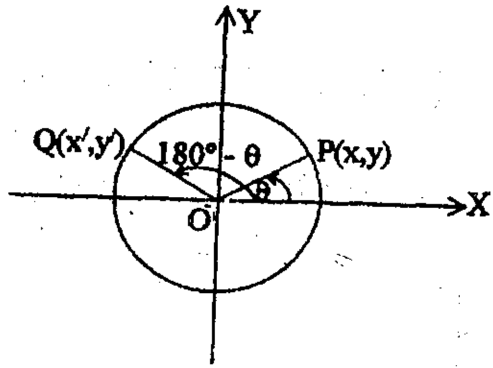


Fig. 11.16

Under a reflection in the Y-axis the point P mapped onto the point Q(x',y').

From Fig. 11.6,

$$\cos(180^\circ - \theta) = x' = -x = -\cos\theta$$

$$\sin(180^\circ - \theta) = y' = y = \sin\theta$$

$$\tan(180^\circ - \theta) = \frac{\sin(180^\circ - \theta)}{\cos(180^\circ - \theta)} = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta$$

We have

$\sin(360^\circ - \theta) = -\sin\theta$, $\quad \sin(180^\circ + \theta) = -\sin\theta$

$\cos(360^\circ - \theta) = \cos\theta$, $\quad \cos(180^\circ + \theta) = -\cos\theta$

$\tan(360^\circ - \theta) = -\tan\theta$, $\quad \tan(180^\circ + \theta) = \tan\theta$

Also,

$\sin(90^\circ + \theta) = \cos\theta \quad \sin(270^\circ - \theta) = -\cos\theta$

$\cos(90^\circ + \theta) = -\sin\theta \quad \cos(270^\circ - \theta) = -\sin\theta$

$\tan(90^\circ + \theta) = -\cot\theta \quad \tan(270^\circ - \theta) = \cot\theta$

11.5 The Basic Acute Angle

The acute angle between the terminal side (i.e. OP) and the X-axis is called the basic acute angle. The basic acute angle is a positive acute angle.

Example 1.

Using **basic acute angle, find** the six trigonometric ratios for the obtuse angle $\frac{2\pi}{3}$ (or) 120°

Solution

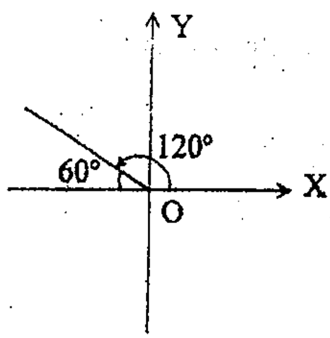


Fig. 11.7

The basic acute angle = 60°

$\cos 120^\circ = \cos 60^\circ$ and $\sin 120^\circ = \sin 60^\circ$ numerically. But 120° is in the second quadrant.

The cosine ratio is negative and the sine ratio is positive.

$$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}, \qquad \sec 120^\circ = -2$$

$$\sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \operatorname{cosec} 120^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}, \quad = -\frac{1}{\sqrt{3}}, \quad \cot 120^\circ = -\sqrt{3}$$

Example 2.

Find $\cos\left(-\frac{\pi}{4}\right)$.

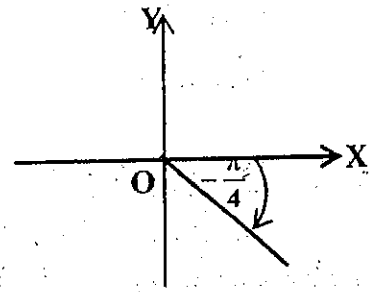


Fig. 11.8

Solution

The basic acute angle $= \frac{\pi}{4}$.

$\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ numerically

But $-\frac{\pi}{4}$ is in the fourth quadrant.

$\cos\left(-\frac{\pi}{4}\right)$ is positive.

$$\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Alternative method

Solution

$$\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}$$

Note: The trigonometric ratio of an angle θ can be found by following steps:

(1) Determine the quadrant in which angle θ is in.

(2) Find the basic acute angle.

(3) The trigonometric ratio of angle θ is equal to the trigonometric ratio of the basic acute angle numerically the sign being determined by using the figure 11.2 diagram.

11.6 Special Angle of 0°, 90°, 180°, 270°, 360°

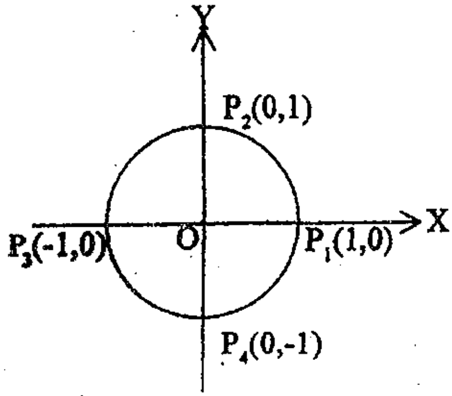


Fig. 11.9

When the terminal side OP makes an angle of $0°$ with the X-axis, P_1 has coordinates(1, 0).

Thus

$\sin 0^\circ = 0,\qquad \cos 0^\circ = 1,\qquad \tan 0^\circ = \dfrac{\sin 0^\circ}{\cos 0^\circ} = \dfrac{0}{1} = 0.$

When the terminal side OP makes an angle of 90° with the X-axis, P_2 has coordinates (0, 1). Thus

$\sin 90^\circ = 1,\qquad \cos 90^\circ = 0,\qquad \tan 90^\circ = \dfrac{\sin 90^\circ}{\cos 90^\circ} = \dfrac{1}{0}$ which is undefined.

When the terminal side OP makes an angle of 180° with the X-axis, P_3 has coordinates $(-1, 0)$. Thus

$\sin 180^\circ = 0,\qquad \cos 180^\circ = -1,\qquad \tan 180^\circ = \dfrac{\sin 180^\circ}{\cos 180^\circ} = \dfrac{0}{-1} = 0.$

When the terminal side OP makes an angle of 270° with the X-axis, P_4 has coordinates $(0, -1)$. Thus

$\sin 270^\circ = -1, \qquad \cos 270^\circ = 0, \qquad \tan 270^\circ = \dfrac{\sin 270^\circ}{\cos 270^\circ} = \dfrac{-1}{0}$ which is undefined.

When the terminal side OP makes an angle of 360° with the X-axis, the trigonometric ratios are exactly the same as those of 0°. Thus

$$\sin 360^\circ = 0, \qquad \cos 360^\circ = 1, \qquad \tan 360^\circ = \frac{\sin 360^\circ}{\cos 360^\circ} = \frac{0}{1} = 0.$$

Exmaple 3.

Find the value of x between 0° and 360° in the following.

(i) $\sin x = \dfrac{1}{2}$ (ii) $\cos x = -\dfrac{1}{2}$ (iii) $\tan \dfrac{1}{2} x = \dfrac{1}{\sqrt{3}}$

Solution

(i) Since $\sin x = \dfrac{1}{2}$, x lies either in the first or second quadrant.

The basic acute angle $= 30^\circ$.

Hence $x = 30^\circ$ (or) $x = 180^\circ - 30^\circ$

$x = 30^\circ$ (or) $x = 150^\circ$

$\therefore x = 30^\circ$ (or) 150°

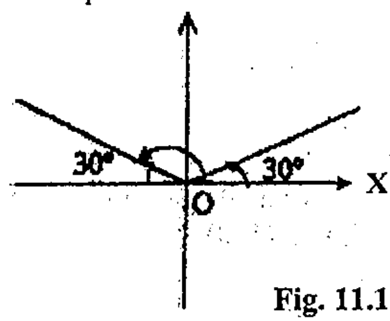


Fig. 11.10

(ii) Since $\cos x = -\dfrac{1}{2}$, x lies either in the second or third quadrant.

The basic acute angle $= 60^\circ$

Hence $x = 180^\circ - 60^\circ$ (or) $180^\circ + 60^\circ$

$x = 120^\circ$ (or) 240°

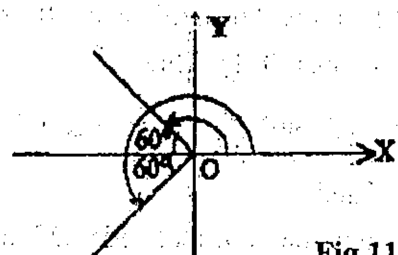


Fig 11.11

(iii) Since $\tan \dfrac{1}{2} x = \dfrac{1}{\sqrt{3}}$, $\dfrac{1}{2}$ x lies either in the first or third quadrant.

The basic acute angle = 30°

$$\text{Hence } \frac{1}{2}x = 30° \quad \text{(or)} \quad 180° + 30°$$

$$x = 60° \quad \text{(or)} \quad 420°$$

$$\therefore x = 60°$$

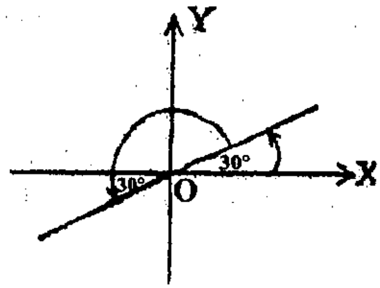


Fig. 11.12.

Exercise 11.1

1. Find the six trigonometric ratios of

   (a) $\theta = \frac{5\pi}{6}$ (b) $\theta = -\frac{\pi}{6}$

   (c) $\theta = -\frac{\pi}{2}$ (d) $\theta = \frac{5\pi}{4}$

2. Find the value of θ, $0 \le \theta < 360°$ for the following equations. Do not use table.

   (a) $\sin\theta = 0$ (b) $\cos\theta = 1$ (c) $\cos\theta = -\frac{1}{2}$

   (d) $\sin\theta = -\frac{\sqrt{3}}{2}$ (e) $\cos\theta = -1$ (f) $\tan\theta = -\sqrt{3}$

   (g) $\sin(2\theta + 15°) = \frac{\sqrt{3}}{2}$ (h) $\tan(3\theta - 30°) = -1$ (i) $\cos 3\theta = -\frac{1}{\sqrt{2}}$

3. Solve the following equations for $0 \le x \le 360°$

   (i) $2\sin x\cos x = \sin x$ (iii) $3\tan x\sin x = 2\tan x$

   (ii) $5\sin x\cos x = 2\cos x$ (iv) $\cos^2 x - \cos x = 2$

4. Find the value of the following. Use table only when necessary.

   (a) $\sec(-150°)$ (b) $\sec(-\pi)$ (c) $\text{cosec}(-\pi)$

   (d) $\tan 0°$ (e) $\cot 150°$ (f) $\tan 120°$

   (g) $\text{cosec}(-80°)$

5. If $\alpha+\beta+\gamma=180^\circ$ prove that

(i) $\sin(\alpha+\beta) = \cos(90^\circ-\gamma)$

(ii) $\sin\frac{\alpha+\beta}{2} = \sin(90^\circ+\frac{\gamma}{2})$

(iii) $\tan\frac{\alpha}{2} = \cot(180^\circ+\frac{\beta+\gamma}{2})$

11.7 Further Trigonometrical Identities

Sum and difference of Two angles

On the unit circle, centre at the origin, let A and B be the points (1, 0) and (0, 1) on axis OX and OY. Let P (x, y) be given by P (x, y) = ($\cos\beta$, $\sin\beta$), so that P corresponds to the angle β ($=\angle AOP$). See Fig. 11.13 (a)

The point P is reached from O by moving a distance x along OA and a distance y parallel to OB, that is

$$(x,y) = x(1,0) + y(0,1)$$

Consider now a rotation of the plane about O, through an angle α in the positive (anticlockwise) direction, inwhich A moves to A' ($\cos\alpha$, $\sin\alpha$) and B moves B' ($-\sin\alpha$, $\cos\alpha$), while P moves to P' [Fig. 11.13 (b)]. Then P' is reached from O by moving a distance x along OA', and a distance y parallel to OB', so that the coordinates (x', y') of P' are given by

$$\begin{aligned}(x', y') &= x(\cos\alpha, \sin\alpha) + y(-\sin\alpha, \cos\alpha)\\ &= (x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)\\ &= (\cos\beta\cos\alpha - \sin\beta\sin\alpha,\ \cos\beta\sin\alpha + \sin\beta\cos\alpha)\end{aligned}$$

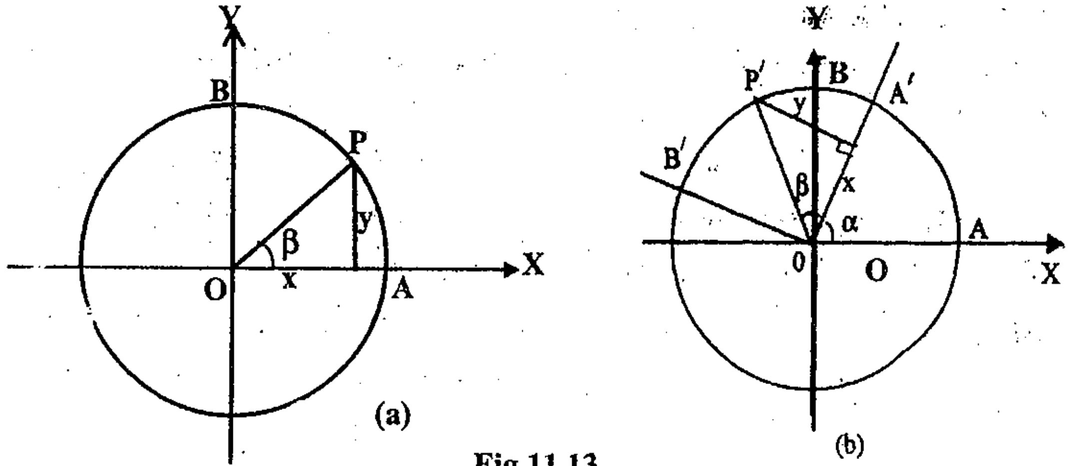


Fig 11.13

But $\angle AOP' = \alpha + \beta$ and so P' represents the angle $\alpha + \beta$ and has coordinates $(\cos(\alpha+\beta), \sin(\alpha+\beta))$.Comparing these two forms for the coordinates of P' ,we find that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

These formulae arc the addition formulae for cosine and sine.

For negative angle, since $\cos(-\beta) = \cos\beta$ and

$$\sin(-\beta) = -\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

The above identities are true for all values of α and β. A summary of the compound angle formulae is given below:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

11.8 Double Angle Formulae

From the formulae for compound angle, more identities can be derived.

Consider

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If $\alpha = \beta$, then $\sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Also,

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 1 - 2\sin^2\alpha \qquad (\because \sin^2\alpha + \cos^2\alpha = 1$$

$$= 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

11.9 Half Angle Formulae

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

Let $t = \tan\frac{\theta}{2}$

Then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

11.10 Factor Formulae

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{\_\_\_\_\_\_(1)}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad \text{\_\_\_\_\_\_(2)}$$

(1) + (2) : $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta$ ______(3)

(1) − (2) : $\sin(\alpha+\beta)-\sin(\alpha-\beta) = 2\cos\alpha\sin\beta$ ______(4)

Let $\theta = \alpha+\beta$ and $\phi = \alpha-\beta$

Then $\theta + \phi = 2\alpha$

$\theta - \phi = 2\beta$

$$\alpha = \frac{\theta+\phi}{2} \quad \text{and} \quad \beta = \frac{\theta-\phi}{2}$$

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

Also, $$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

The factor formulae can be used to simplify the sum and differences of the sine and cosine of two angles. In addition, they are useful when differentiating trigonometrical ratios.

11.11 Equations of the Type a cos θ + b sin θ = c

$$\left.\begin{aligned} \text{Let} \quad a &= R\cos\alpha \\ \text{and} \quad b &= R\sin\alpha \end{aligned}\right\} \text{---------------(1)}$$

Then $R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = c$

$$R\cos(\theta-\alpha) = c \text{ ---------------(2)}$$

From (1) we get $a^2+b^2 = R^2(\cos^2\alpha + \sin^2\alpha)$

$$R^2 = a^2+b^2 \qquad (\because \cos^2\alpha + \sin^2\alpha = 1)$$

$$R = \pm\sqrt{a^2+b^2}$$

$$\text{and } \tan\alpha = \frac{b}{a}$$

Taking only the positive root, equation (2) becomes,

$$\sqrt{a^2+b^2}\ \cos(\theta-\alpha)=c.$$

We have

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\ \cos(\theta-\alpha) \text{ where } \tan\alpha = \frac{b}{a}, \text{ a and b}$$

are positive and α is acute.

Similarly, $a\cos\theta - b\sin\theta = \sqrt{a^2+b^2}\ \cos(\theta+\alpha)$ where $\tan\alpha = \frac{b}{a}$, a and b are positive and α is acute.

Again, $a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\ \sin(\theta+\alpha)$ where $\tan\alpha = \frac{b}{a}$, a and b are positive and α is acute.

And $a\sin\theta - b\cos\theta = \sqrt{a^2+b^2}\ \sin(\theta-\alpha)$ where $\tan\alpha = \frac{b}{a}$, a and b are positive and α is acute.

11.12 Proving of Identities

We have learnt that a trigonometrial identity is an equation or expression that holds true for all values of the angles involved in the expression. For instance, $1+\tan^2\alpha = \sec^2\alpha$ is true for all values of α, we shall use the sign '$\equiv$' for identities in the following example and exercises. By using the various trigonometircal ratios and identities so far derived, we can easily convert trigonometrical expressions into different forms.

Normally, exercises on trigonometrical identities give two expressions that must be proved equal. There is no general approach to the type of exercises, but the following guide lines may be used.

(a) The approach is usually to substitute identities to make the left-hand side of the identity equal to its right-hand side.

(b) Sometimes the problem is made easier by simply rearranging the given identity. In gencral, tackle the more complicated expression by first simplifying it.

(c) If say, after 8-10 steps, the required result is still unobtainable with the substitution, try a different substitution.

(d) Look for expressions like $\sin^2 \alpha + \cos^2 \alpha = 1$.

$\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$ and $\cot \alpha = \dfrac{\cos \alpha}{\sin \alpha}$ whenever necessary.

(e) It is sometimes useful to reduce every expression to $\sin \alpha$ and $\cos \alpha$.

Example 1.

Express the following as single trigonometric ratios:

(i) $\sin 37^\circ \cos 41^\circ + \cos 37^\circ \sin 41^\circ$ (ii) $\dfrac{1-\tan 15^\circ}{1+\tan 15^\circ}$

Solution

(i) $\sin 37^\circ \cos 41^\circ + \cos 37^\circ \sin 41^\circ = \sin(37^\circ + 41^\circ) = \sin 78^\circ$

(ii) $\dfrac{1-\tan 15^\circ}{1+\tan 15^\circ} = \dfrac{\tan 45^\circ - \tan 15^\circ}{1+\tan 45^\circ \tan 15^\circ} = \tan(45^\circ - 15^\circ) = \tan 30^\circ$

($\because \tan 45^\circ = 1$)

Example 2.

Find without using table, the value of (i) $\sin 75^\circ$ and (ii) $\tan 15^\circ$.

Solution

(i) $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \dfrac{\tan 60^\circ - \tan 45^\circ}{1+\tan 60^\circ \tan 45^\circ} = \dfrac{\sqrt{3}-1}{1+(\sqrt{3})(1)}$

$$= \left(\frac{\sqrt{3}-1}{1+\sqrt{3}}\right) \times \left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right) = \frac{2\sqrt{3}-4}{-2} = 2 - \sqrt{3}$$

Example 3.

Given that $\sin\alpha = -\dfrac{4}{5}$, $\cos\beta = -\dfrac{12}{13}$ and that α and β are in the same quadrant, find each of the following without the use of tables.

(i) $\sin 2\alpha$ (ii) $\cos 2\alpha$ (iii) $\cos\dfrac{\beta}{2}$ (iv) $\tan 2\beta$

Solution

$\sin\alpha$ and $\cos\beta$ are negative and must be in the same quadrant, α and β lies in the third quadrant. From the Fig. 11.14

$$\cos\alpha = -\frac{3}{5}$$

(i) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{24}{25}$$

(ii) $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$= 1 - 2\left(-\frac{4}{5}\right)^2 = -\frac{7}{25}$$

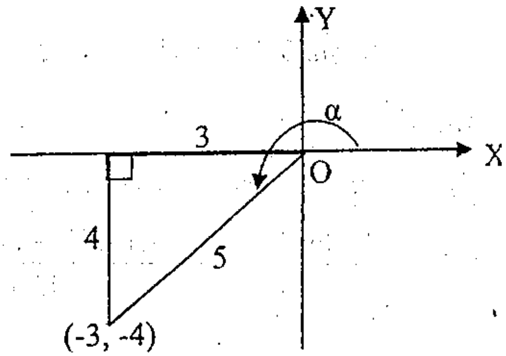


Fig. 11.14

From the Fig. 11.15

$\sin\beta = -\dfrac{5}{13}$ $\tan\beta = \dfrac{5}{12}$

(iii) $\cos\beta = 2\cos^2\dfrac{\beta}{2} - 1$

$$-\frac{12}{13} = 2\cos^2\frac{\beta}{2} - 1$$

$$\cos^2\frac{\beta}{2} = \frac{1}{26}$$

$$\cos\frac{\beta}{2} = \pm\sqrt{\frac{1}{26}}$$

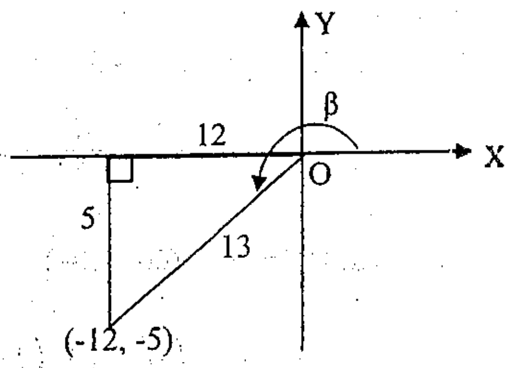


Fig. 11.15

$180^\circ < \beta < 270^\circ$, we have $90^\circ < \frac{\beta}{2} < 135^\circ$, i.e. $\frac{\beta}{2}$ lies in the second quadrant.

$$\therefore \qquad \cos\frac{\beta}{2} = -\sqrt{\frac{1}{26}} = -\frac{\sqrt{26}}{26}$$

(iv) $$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2(\frac{5}{12})}{1-(\frac{5}{12})^2} = \frac{120}{119}$$

Example 4.

If $\sin\theta = a$, where θ is an acute angle express the following in terms of a:

(i) $\tan^2\theta$ (ii) $\cos 2\theta$ **(iii) $\sin 4\theta$** (iv) $\cos^2\frac{1}{2}\theta$

Solution

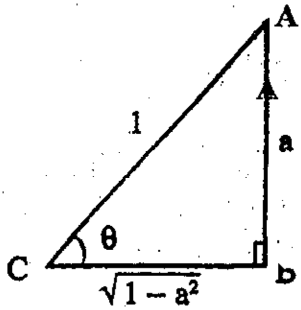


Fig. 11.16

In the right-angled triangle ABC, let AB = a and AC = 1 then BC = $\sqrt{1-a^2}$.

(i) $$\tan\theta = \frac{a}{\sqrt{1-a^2}}$$

$$\tan^2\theta = \left(\frac{a}{\sqrt{1-a^2}}\right)^2 = \frac{a^2}{1-a^2}$$

(ii) $$\cos 2\theta = 1-2\sin^2\theta = 1-2a^2$$

(iii) $$\sin 4\theta = \sin 2(2\theta) = 2\sin 2\theta\cos 2\theta$$
$$= 2(2\sin\theta\cos\theta)\cos 2\theta$$
$$= 2(2a\sqrt{1-a^2})(1-2a^2) = 4a(1-2a^2)(\sqrt{1-a^2})$$

(iv) $$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$2\cos^2\frac{\theta}{2} = 1+\cos\theta = 1+\sqrt{1-a^2}$$

$$\cos^2\frac{\theta}{2} = \frac{1+\sqrt{1-a^2}}{2}$$

Example 5.

Express the following as factors.

(i) $\sin 5x + \sin 3x$

(ii) $\cos 3x - \cos 5x$

Solution

(i) $\sin 5x + \sin 3x = 2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2} = 2\sin 4x\cos x$

(ii) $\cos 3x - \cos 5x = -2\sin\frac{3x+5x}{2}\sin\frac{3x-5x}{2} = -2\sin 4x\sin(-x)$

$$= 2\sin 4x\sin x \quad (\because \sin(-x) = -\sin x)$$

Example 6.

Express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta-\alpha)$ where R and θ are constants. Hence solve the equation $4\cos\theta + 3\sin\theta = 2$ for value of θ between 0° and 360°.

Solution

Let $4\cos\theta + 3\sin\theta = R\cos(\theta-\alpha)$ where $R = \sqrt{4^2+3^2} = 5$

and $\tan\alpha = \frac{b}{a} = \frac{3}{4} = 0.75 = \tan 36^\circ\ 52'$

$$\alpha = 36^\circ\ 52'$$

$\therefore 4\cos\theta + 3\sin\theta = 5\cos(\theta - 36^\circ\ 52')$

Hence $5\cos(\theta-36^\circ\ 52') = 2$

$$\therefore \cos(\theta-36^\circ\ 52') = \frac{2}{5} = 0.4 = \cos 66^\circ\ 25'$$

$$\theta-36^\circ\ 52' = 66^\circ\ 25' \text{ (or) } 360^\circ - 66^\circ\ 25' = 66^\circ\ 25' \text{ (or) } 293^\circ\ 35'$$

$$\theta = 103^\circ\ 17' \text{ (or) } 330^\circ\ 27'$$

Example 7.

Prove that $\tan\alpha + \cot\alpha = \dfrac{2}{\sin 2\alpha}$.

Solution

$$\tan\alpha + \cot\alpha = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha} \qquad = \frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{1}{\sin\alpha\cos\alpha} \qquad (\because \sin^2\alpha + \cos^2\alpha = 1)$$

$$= \frac{2}{2\sin\alpha\cos\alpha} \qquad = \frac{2}{\sin 2\alpha}$$

Example 8.

Prove that $\operatorname{cosec}\theta = \dfrac{\cos 2\theta}{\sin\theta} + \dfrac{\sin 2\theta}{\cos\theta}$.

Solution

$$\text{R.H.S} = \frac{\cos 2\theta}{\sin\theta} + \frac{\sin 2\theta}{\cos\theta} \qquad = \frac{\cos 2\theta\cos\theta + \sin 2\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos(2\theta - \theta)}{\sin\theta\cos\theta} \qquad = \frac{\cos\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta}$$

$$= \operatorname{cosec}\theta \qquad = \text{L.H.S}$$

Exercise 11.2

1. Express the following as single trigonometric ratios.
   (i) $\cos 25^\circ \cos 15^\circ - \sin 25^\circ \sin 15^\circ$
   (ii) $\cos 75^\circ \cos 24^\circ + \sin 75^\circ \sin 24^\circ$
   (iii) $\sin 126^\circ \cos 23^\circ - \cos 126^\circ \sin 23^\circ$
   (iv) $\dfrac{\tan 27^\circ + \tan 13^\circ}{1 - \tan 27^\circ \tan 13^\circ}$
2. Use the compound angle formulae to find the following in surd form:
   (i) $\sin 105^\circ$ (ii) $\cos 15^\circ$ (iii) $\tan 75^\circ$ (iv) $\cos 165^\circ$
   (v) $\sin 345^\circ$ (vi) $\tan 225^\circ$

3. Given that $\sin\alpha = \frac{15}{17}$ and that $\cos\beta = -\frac{3}{5}$ and that α and β are in the same quadrant, find without using tables, the value of

(i) $\sin 2\alpha$ (ii) $\cos\frac{1}{2}\alpha$ (iii) $\cos 2\beta$

4. (i) Express $\cos 3x$ in terms of $\cos x$.

(ii) Express $\sin 3x$ in terms of $\sin x$.

5. Given that α is acute and $\cos\alpha = x$, find, in terms of x, the value of

(i) $\tan^2\alpha$ (ii) $\sin 2\alpha$ (iii) $\cos 4\alpha$ (iv) $\sin\frac{1}{2}\alpha$.

6. Express the following as factors.

(i) $\sin 3\alpha + \sin\alpha$ (ii) $\sin 5\alpha - \sin\alpha$

(iii) $\cos 2\alpha + \cos 7\alpha$ (iv) $\cos 9\alpha - \cos\alpha$

7. Show that $\sin\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{2-\sqrt{3}}{4}$.

8. If $\alpha + \beta + \gamma = \pi$ show that

(i) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

(ii) $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

9. Without the use of table evaluate $\tan(\alpha+\beta+\gamma)$, given that $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$ and $\tan\gamma = \frac{1}{4}$.

10. Find the exact value of $\cos 15^\circ$ given that $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

11. Given that $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{5}{2}$, show that $3\tan\alpha = 7\tan\beta$.

Given further that $\alpha+\beta = 45^\circ$, find the value of $\tan\alpha + \tan\beta$.

12. Given that $\sin\alpha = \frac{5}{13}$, where $90^\circ < \alpha < 180^\circ$ and that $\cos\beta = -\frac{3}{5}$, where $180^\circ < \beta < 360^\circ$, find the values of

(a) $\tan(\alpha + 45^\circ)$ (b) $\sin(\alpha+\beta)$ (c) $\cos 2\alpha$ (d) $\sin 2\beta$.

13. Solve the equation $3\cos\theta - 2\sin\theta = 2$ for values of θ between 0° and 360°.

14. Prove that (i) $\dfrac{\tan y - \tan x}{\tan y + \tan x} = \dfrac{\sin(y-x)}{\sin(y+x)}$

(ii) $\cos(60^\circ + x) + \sin(30^\circ + x) = \cos x$.

11.13 The Law of Cosines and The Law of Sines

There are some important relationships between the parts of a triangle. We will study two of these, namely, the law of cosines and the law of sines.

The Law of Cosines

If ΔABC is an arbitrary triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively, then

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = c^2 + a^2 - 2ca\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

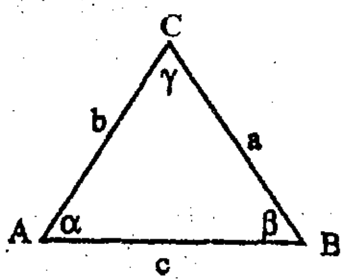


Fig. 11.17

Proof:

Case(i) $\angle A$ is acute angle.

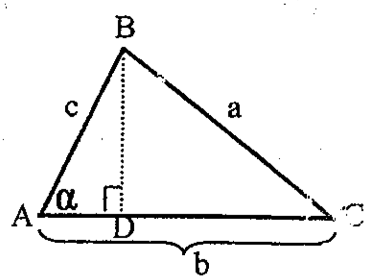


Fig 11.18

In right ΔADB, $\dfrac{AD}{AB} = \cos\alpha$

$$AD = AB\cos\alpha$$
$$= c\cos\alpha$$

In right ΔADB, $BD^2 = AB^2 - AD^2$

In right ΔBDC, $BD^2 = BC^2 - DC^2$

Therefore, $BC^2 - DC^2 = AB^2 - AD^2$

$$BC^2 = DC^2 + AB^2 - AD^2$$
$$BC^2 = (AC - AD)^2 + AB^2 - AD^2$$

$$BC^2 = AC^2 - 2AC \,.\, AD + AD^2 + AB^2 - AD^2$$
$$BC^2 = AC^2 + AB^2 - 2AC \,.\, AD$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Case(ii) $\angle A$ is obtuse angle.

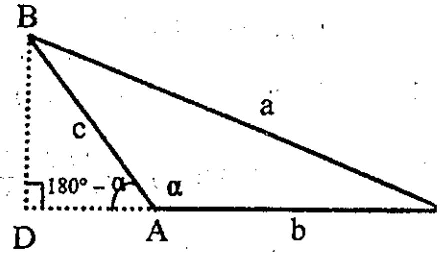


Fig 11.19

In right ΔADB, $\dfrac{AD}{AB} = \cos(180° - \alpha)$

$$AD = AB\cos(180° - \alpha)$$
$$= -c\cos\alpha$$

In right ΔADB, $BD^2 = AB^2 - AD^2$

In right ΔBDC, $BD^2 = BC^2 - DC^2$

Therefore, $BC^2 - DC^2 = AB^2 - AD^2$

$$BC^2 = DC^2 + AB^2 - AD^2$$
$$BC^2 = (AC + AD)^2 + AB^2 - AD^2$$
$$BC^2 = AC^2 + 2AC \,.\, AD + AD^2 + AB^2 - AD^2$$
$$BC^2 = AC^2 + AB^2 + 2AC \,.\, AD$$
$$a^2 = b^2 + c^2 + 2b(-c\cos\alpha)$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Similarly, we can prove that $b^2 = c^2 + a^2 - 2ca\cos\beta$

and $c^2 = a^2 + b^2 - 2ab\cos\gamma$

The Law of Cosines is used when we are given either

(1) two sides of a triangle and the angle between them (included angle)

(or) (2) three sides of a triangle.

The Law of Sines

Consider an arbitrary triangle with angles α, β, γ and corresponding opposite sides a, b, c respectively. Then

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Equivalently $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}.$

Proof:

Case(i) $\angle A$ is acute angle.

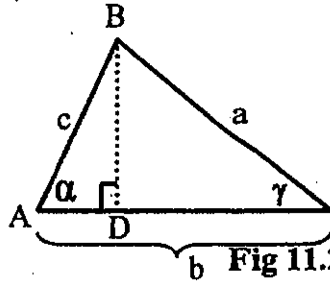


Fig 11.20

In right $\triangle ADB$, $\dfrac{BD}{AB} = \sin\alpha$

$$BD = AB\sin\alpha = c\sin\alpha$$

In right $\triangle BDC$, $\dfrac{BD}{BC} = \sin\gamma$

$$BD = BC\sin\gamma = a\sin\gamma$$

Therefore, $a\sin\gamma = c\sin\alpha$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

Case(ii) $\angle A$ is obtuse angle.

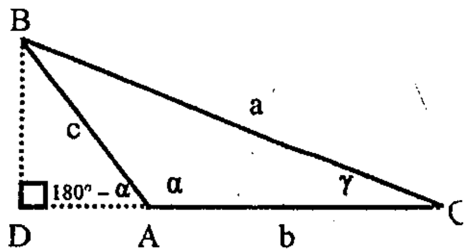


Fig. 11.21

In right $\triangle ADB$, $\dfrac{BD}{AB} = \sin(180° - \alpha)$

$$BD = AB\sin(180° - \alpha)$$
$$= c\sin\alpha$$

In right $\triangle BDC$, $\dfrac{BD}{BC} = \sin\gamma$

$$BD = BC\sin\gamma = a\sin\gamma$$

Therefore, $a\sin\gamma = c\sin\alpha$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

Similarly, we can prove that $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

Hence, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$.

The Law of Sines is used when we are given

either (1) two angles and one side

(or) (2) two sides and an opposite angle.

Example 1.

Solve ΔABC with $b = 18.1$, $c = 12.3$ and $\alpha = 115^\circ$.

Solution

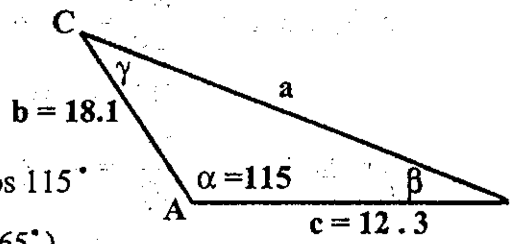


Fig 11.22

By the Law of Cosines,

$$a^2 = (18.1)^2 + (12.3)^2 - 2\,(18.1)\,(12.3)\cos 115^\circ$$
$$= 327.6 + 151.3 - (18.1)\,(24.6)\,(-\cos 65^\circ)$$
$$= 478.9 + (18.1)\,(24.6)\cos 65^\circ$$
$$= 478.9 + 188.1$$
$$= 667$$
$$a = 25.83$$

no	log
18.1	1.2577
24.6	1.3909 +
cos 65°	$\bar{1}.6259$
188.1 ←	2.2745

By the Law of Sines,

$$\frac{\sin\gamma}{12.3} = \frac{\sin 115^\circ}{25.83}$$

$$\sin\gamma = \frac{12.3\sin 65^\circ}{25.83}$$
$$= \sin 25^\circ 34'$$
$$\gamma = 25^\circ 34'$$
$$\alpha + \beta + \gamma = 180^\circ$$
$$\beta = 180^\circ - (115^\circ + 25^\circ 3') = 180^\circ - 140^\circ\ 34'$$
$$= 39^\circ\ 26'$$

no	log
12.3	1.0899
sin 65°	$\bar{1}.9573$ +
	1.0472
25.83	1.4121 −
sin 25° 34' ←	$\bar{1}.6351$

Example 2.

In ΔABC, $a = 5$, $\beta = 75^\circ$ and $\gamma = 41^\circ$. Find α, b and c.

Solution

$$\alpha+\beta+\gamma = 180^\circ$$

$$\alpha = 180^\circ - (\beta+\gamma) = 180^\circ - (75^\circ + 41^\circ)$$

$$= 180^\circ - 116^\circ = 64^\circ$$

To find the side b.

We will use the Law of Sines,

no	log
5	0.6990
sin 75°	$\bar{1}.9849$
cosec 64°	0.0463
5.373 ←	0.7302

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$$

$$b = \frac{a\sin\beta}{\sin\alpha}$$

$$= \frac{5\sin 75^\circ}{\sin 64^\circ} = 5\sin 75^\circ \operatorname{cosec} 64^\circ$$

$$= 5.373$$

We can solve for c in the same manner

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$c = \frac{a\sin\gamma}{\sin\alpha} = \frac{5\sin 41^\circ}{\sin 64^\circ}$$

$$= 5\sin 41^\circ \operatorname{cosec} 64^\circ$$

$$= 3.65$$

no	log
5	0.6990
sin 41°	$\bar{1}.8169$
cosec 64°	0.0463
3.65 ←	0.5622

Example 3.

Solve ΔABC with $a = 3$, $b = 4$, $c = 6$.

Solution

Any triangle the longest side lies opposite to the largest angle and the shortest side lies opposite to the smallest angle. In the example, we have $\alpha < \beta < \gamma$.

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{4^2+6^2-3^2}{2(4)(6)}$$

$$= \frac{16+36-9}{48}$$

no	log
43	1.6335
48	1.6812
cos 26° 23' ←	$\bar{1}.9523$

$$= \frac{43}{48}$$

$$\cos \alpha = \cos 26^\circ \; 23'$$

$$\alpha = 26^\circ \; 23'$$

We can solve for β in the same manner.

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{3^2 + 6^2 - 4^2}{2(3)(6)}$$

$$= \frac{9 + 36 - 16}{36}$$

$$= \frac{29}{36}$$

no	log
29	1.4624
36	1.5563
cos 36° 20' ←	$\bar{1}.9061$

$$\cos \beta = \cos 36^\circ \; 20'$$

$$\beta = 36^\circ \; 20'$$

$$\alpha + \beta + \gamma = 180^\circ$$

$$\gamma = 180^\circ - (\alpha + \beta) = 180^\circ - (26^\circ \; 23' + 36^\circ \; 20')$$

$$= 180^\circ - 62^\circ \; 43'$$

$$= 117^\circ \; 17'$$

11.14 Bearings

The four cardinal directions are North, South, East and West Fig. 11.23. The direction NE, NW, SE and SW are frequently used and are as shown in .

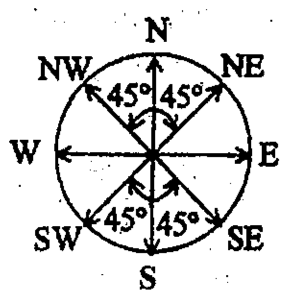


Fig. 11.23

A bearing of N 20° E means an angle of 20° measured from the N towards E as shown in Fig. 11.23.

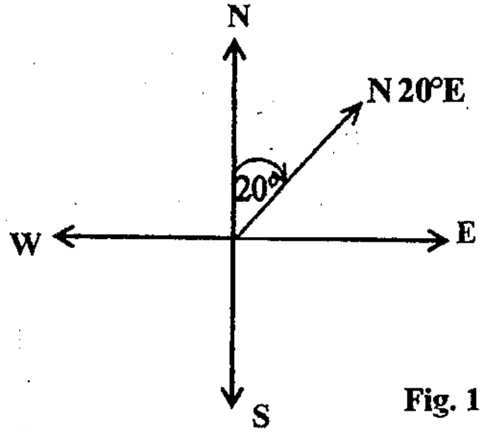


Fig. 11.24

Similarly a bearing of S 40° E means an angle 40° measured from the S towards E (Fig. 11.25)

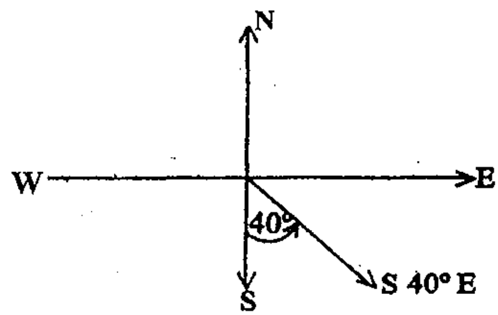


Fig 11.25

A bearing of N 50° W means an angle of 50° measured from N towards W.

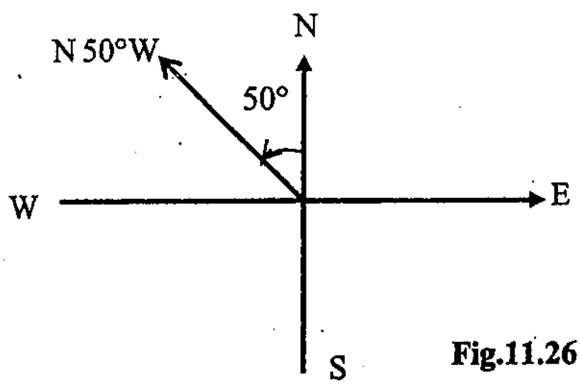


Fig.11.26

(Fig, 11.26) Bearing quoted in this way are always measured from N and S and never from E and W.

There is a second way of starting a bearing. The angle denoting the bearing is measured from N in a clockwise direction, N being reckoned as 0° . Three figures are always stated, for example 005° is written instead of 5° , 035° for 35° etc. East will be 90°, South 180° and West 270°. Some typical bearings are shown in Fig. 11.27.

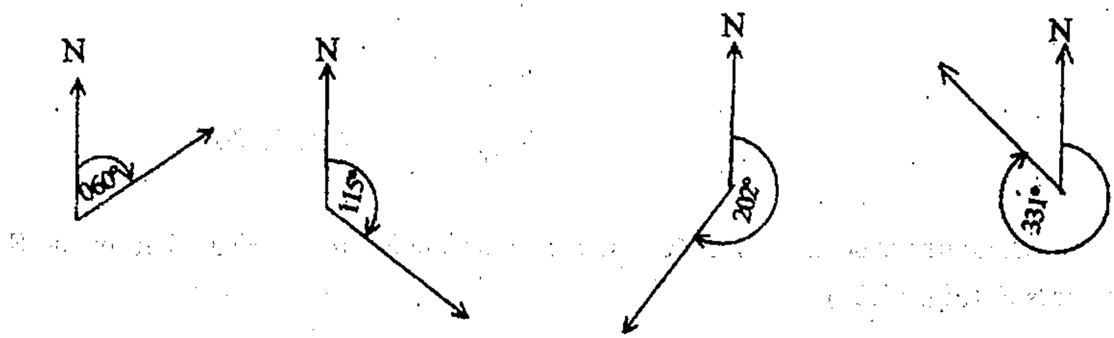


Fig. 11.27

Example 4.

A man travels 10 km in a direction N 80° E and then 5 km in a direction N 40° E. What is his final distance and bearing from his starting point?

Solution

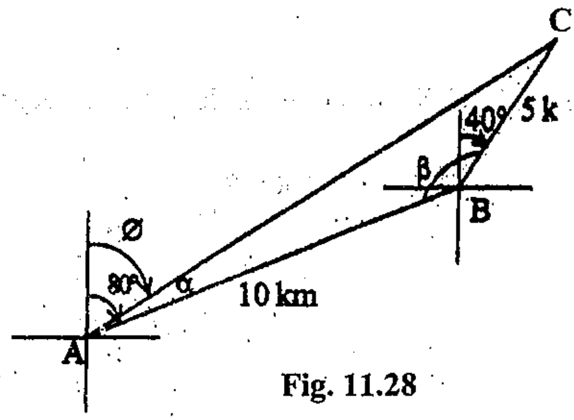


Fig. 11.28

$c = 10$ km, $a = 5$ km, $\beta = 140°$

By the Law of Cosines

$$b^2 = 100 + 25 - 2(10)(5)\cos 140^\circ$$

$$= 125 - 100(-\cos 40^\circ) = 125 + 100\cos 40^\circ$$

$$= 125 + 76.6 = 201.6$$

$$b = 14.2 \text{ km}$$

The distançe from the starting point is 14.2 km.

By the Law of Sines

$$\frac{\sin\alpha}{5} = \frac{\sin 140^\circ}{14.2}$$

$$\sin\alpha = \frac{5\sin 40^\circ}{14.2}$$

$$= \sin 13^\circ\ 5'$$

$$\alpha = 13^\circ\ 5'$$

$$\phi = 80^\circ - \alpha = 80^\circ - 13^\circ\ 5' = 66^\circ\ 55'$$

no	log
5	0 . 6990
sin40°	$\bar{1}$. 8081 +
	0 . 5071
14.2	1 . 1523 −
sin13°5′	$\bar{1}$. 3548

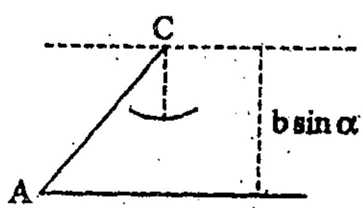
∴ It is in the direction N 66° 55' E.

<u>Ambigious Case</u> The last case when two sides and an angle opposite to one sid. are given is called the ambigious case because there may be no triangle, one triangle, or two triangles satisfying the given conditions.

The diagrams below illustrate the various possibilities given A, b and a. They are divided into two cases.

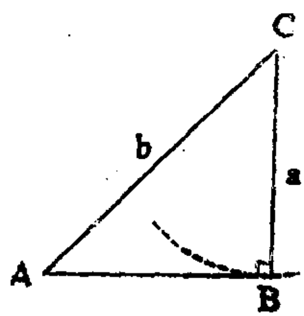
$\alpha < 90^\circ$ and $\alpha \geq 90^\circ$

Case I: $\alpha < 90^\circ$



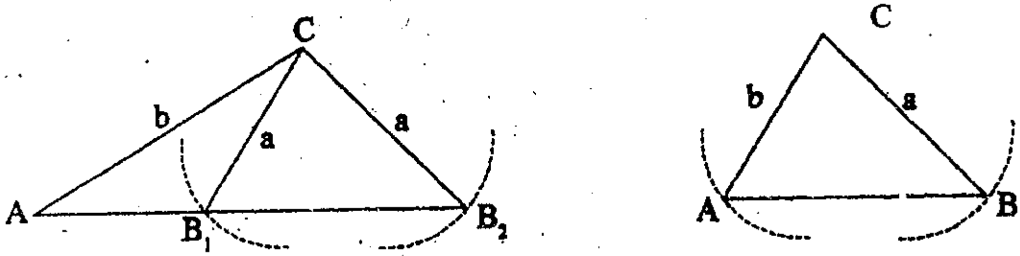
(i)

Fig. 11.29



(ii)

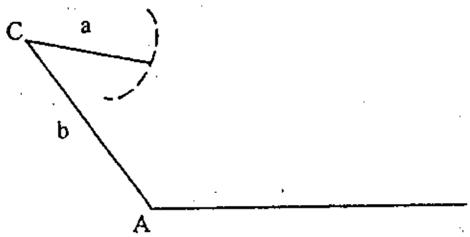
No solution: $a < b \sin \alpha$ One solution: $a = b \sin \alpha$

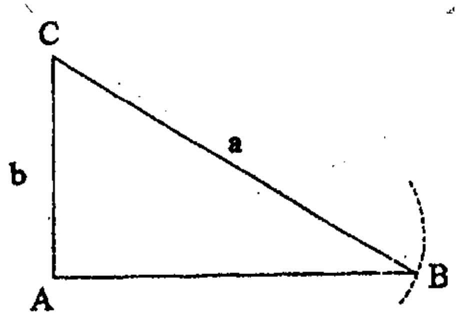


(iii) **Fig. 11.30** (iv)

Two solutions: $b \sin \alpha < a < b$ **One solution: $a > b$**

Case II: $\alpha \geq 90^\circ$





(i) **Fig. 11.31** (ii)

No solution: $a \leq b$ **One solution: $a > b$**

The Law of Sines can be used to solve a triangle for which the data are ambigious. However, it is important to sketch and label the triangle first. Then determine the number of possible solutions.

Example 5.

Find the number of solutions for each triangle.

(a) $\alpha = 30^\circ$ (b) $\alpha = 30^\circ$ (c) $\alpha = 30^\circ$

$a = 6$ $a = 8$ $a = 4$

$b = 12$ $b = 12$ $b = 12$

Solution

(a) $b \sin \alpha = 12 \sin 30^\circ$

$= 12 (\frac{1}{2})$

$= 6$

Therefore $b \sin \alpha = a$

$\therefore$ One solution.

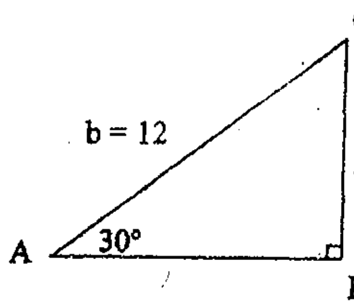


Fig. 11.32

(b) $b \sin \alpha = 12 \sin 30^\circ$

$= 12 (\frac{1}{2})$

$= 6$

$b \sin \alpha < a < b$

$\therefore$ Two solutions.

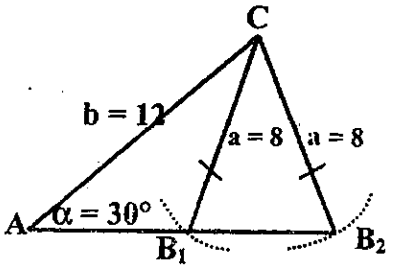


Fig. 11.33

(c) $b \sin \alpha = 12 \sin 30^\circ$

$= 12 (\frac{1}{2}) = 6$

$a < b \sin \alpha$

$\therefore$ No solution.

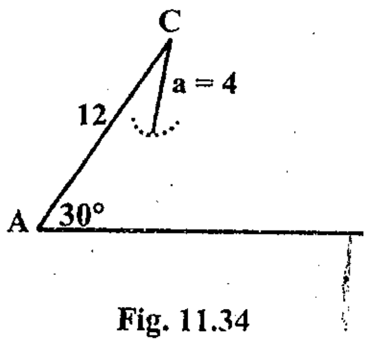


Fig. 11.34

When there two solutions, you must solve both triangles.

Example 6.

In ΔABC, $\alpha = 30^\circ$, $a = 15$, $b = 20$. Solve the triangle.

Solution

$b \sin \alpha = 20 \sin 30^\circ$

$= 20 (\frac{1}{2}) = 10$

$b \sin \alpha < a < b$

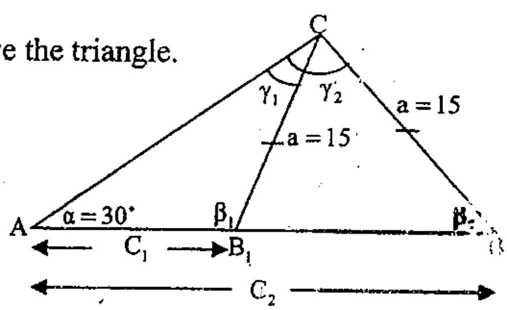


Fig. 11.35

Thus there tow solutions, corresponding to two triangles AB_1C and AB_2C.

First solve ΔAB_1C.

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin\beta_1}$$

$$\frac{15}{\frac{1}{2}} = \frac{20}{\sin\beta_1}$$

$$\sin\beta_1 = \frac{20}{30} = \frac{2}{3} = 0.6667$$

$$= \sin 41^\circ\ 49'$$

$$\beta_1 = 41^\circ 49'$$

Since β_1 is obtuse,

$$\beta_1 = 180^\circ - 41^\circ\ 49' = 138^\circ\ 11'$$

$$\gamma_1 = 180^\circ - (138^\circ\ 11' + 30^\circ) = 11^\circ\ 49'$$

Again by the Law of Sines

$$\frac{c_1}{\sin\gamma_1} = \frac{a}{\sin\alpha}$$

$$c_1 = \frac{15\sin 11^\circ\ 49'}{\sin 30^\circ}$$

$$= 15\sin 11^\circ\ 49' \operatorname{cosec} 30^\circ = 6.144$$

no	log
15	1.1761
sin 11° 49′	$\bar{1}.3113$
cosec 30°	0.3010
6.144 ←	0.7884

To solve ΔAB_2C.

It is clear from the figure, ΔB_1CB_2 is isosceles

$$\beta_2 = 180^\circ - \beta_1 = 180^\circ - 138^\circ\ 11' = 41^\circ 49'$$

$$\gamma_2 = 180^\circ - (30^\circ + 41^\circ\ 49') = 108^\circ\ 11'$$

$$\frac{c_2}{\sin\gamma_2} = \frac{15}{\sin 30^\circ}$$

no	log
15	1.1761
sin 71°49′	$\bar{1}.9777$
cosec30°	0.3010
28.49 ←	1.4548

$$c_2 = \frac{15\sin 108^\circ 11'}{\sin 30^\circ}$$

$$= 15 \sin 108^\circ 11' \operatorname{cosec} 30^\circ$$

$$= 15 \sin 71^\circ 49' \operatorname{cosec} 30^\circ = 28.49$$

Exercise 11.3

1. Find a if $b = 4$, $c = 11$ and $\alpha = 60^\circ$
2. Find b if $a = 20$, $c = 8$ and $\beta = 45^\circ$.
3. Find c if $\gamma = 30^\circ$, $\alpha = 135^\circ$ and $a = 100$.
4. Find a and c if $\alpha = 30^\circ$, $\beta = 120^\circ$ and $b = 54$.
5. Find γ if $a = 12$, $b = 5$ and $c = 13$.
6. In ΔABC, $c = 20$, $b = 12$ and $a = 10$. Check whether $\angle ACB$ is acute or obtuse and find its magnitude.

Solve the following triangles.

7.	$\alpha = 25^\circ$,	$\gamma = 55^\circ$	b 12
8.	$\gamma = 110^\circ$,	$\beta = 28^\circ$,	$a = 8$
9.	$a = 9$,	$b = 11$,	$\gamma = 60^\circ$
10.	$a = 5$,	$b = 8$,	$c = 7$
11.	$\angle A = 64^\circ 20'$,	$\angle B = 50^\circ$,	$b = 5$
12.	$\angle A = 154^\circ$,	$\angle B = 15^\circ 30'$,	$c = 20$
13.	$\angle A = 53^\circ$,	$a = 12$,	$b = 15$.

14. A man standing at a point P, sees two trees, X and Y, which are respectively 250m and 310 m away from him. If $\angle XPY = 120^\circ$ how far apart are the two trees.
15. A ship leaves harbour on a course N 72° E, and after travelling for 50 metres, changes course to 108°. After a further 106 metres,

    find (a) the distance of the ship from the harbour

    (b) its bearing from the harbour.

16. To approximate the distance between two points A and B on opposite sides of a swamp, a surveyour selects a point C and measures it to be 140 metres from A and 260 metres form B. Then he measures the angle ACB, which turns out to be 49°. What is the distance from A to B?

17. Two runners start from the same point at 12 : 00 noon, one of them heading north at 6 m.p.h and the other heading 68° east of north at 8 m.p.h. What is the distance between them at 3 : 00 that afternoon?

18. In ΔABC, $AB = x$, $BC = x + 2$ and $AC = x - 2$ where $x > 4$, prove that $\cos A = \dfrac{x-8}{2(x-2)}$.

Find the integral values of x for which A is obtuse..

19. ΔABC is an acute triangle .Prove that $\tan \alpha = \dfrac{a \sin\gamma}{b - a\cos\gamma}$.

20. A, B, C are three towns, B is 10 miles from A in a direction N 47° E. C is 17 miles away from B in a direction N 70° W. Calculate the distance and direction of A from C.

21. A ship is 5 km away from a boat in a direction N 37° W and a lighthouse is 12 km away from the boat in a direction S 53° W. Calculate the distance and direction of the ship from the lighthouse.

22. A road PQ of length 725 km runs straight from P to East. The bearings of a school C from P and Q are 042° and 325° respectively . What is the distance of the school from the road?

23. A town P is 50 miles away from a town Q in the direction N 35° E and a town R is 68 miles from Q in the direction N 42° 12′ W. Calculate the distance and bearing of P from R.

11.15 Graphs of sin x, cos x and tan x

By plotting the value of x on the X-axis and the values of sinx on the Y-axis, the sine curve (see Fig. 11.36) is obtained. Notice that the maximum and the minimum value at sinx are 1 and -1 respectively.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Table 11.2

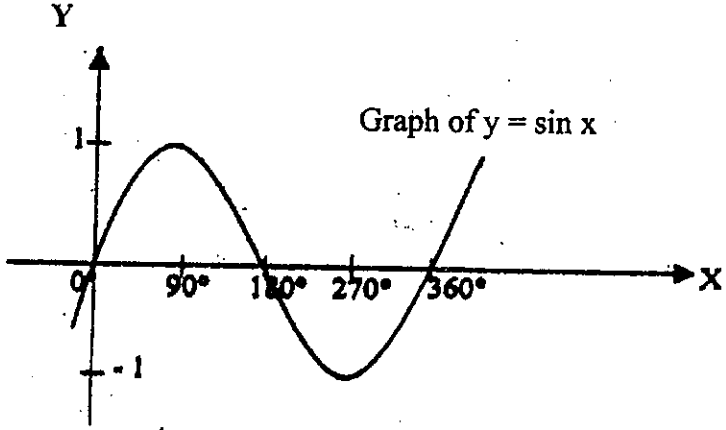


Fig. 11.36

The cosine and tangent curve are plotted accordingly. Fig. 11.37 and Fig. 11.38 show the cosine curve and tangent curve respectively.

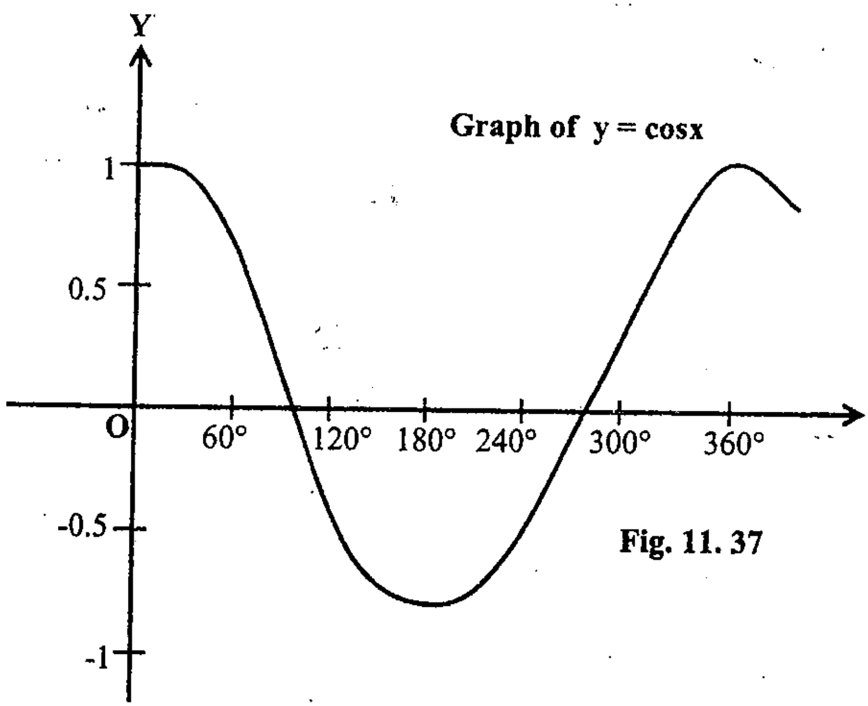


Fig. 11. 37

The two graphs above show that the sine and cosine curves are similar in shape. There is a complete cycle for $0^\circ \leq x \leq 360^\circ$ in both the sine and cosine graphs. The only difference is that the cosine curve is lagging 90° behind the sine curve.This 90 difference is called the phase difference. For angles greater than 360° or less than 0° the curve respect themselves in periodic cycles. Such functions are called periodic function. The sine and cosine functions are examples of periodic functions each with a period 360°. The sine and cosine curves are useful especially in the study of waves and electricity.

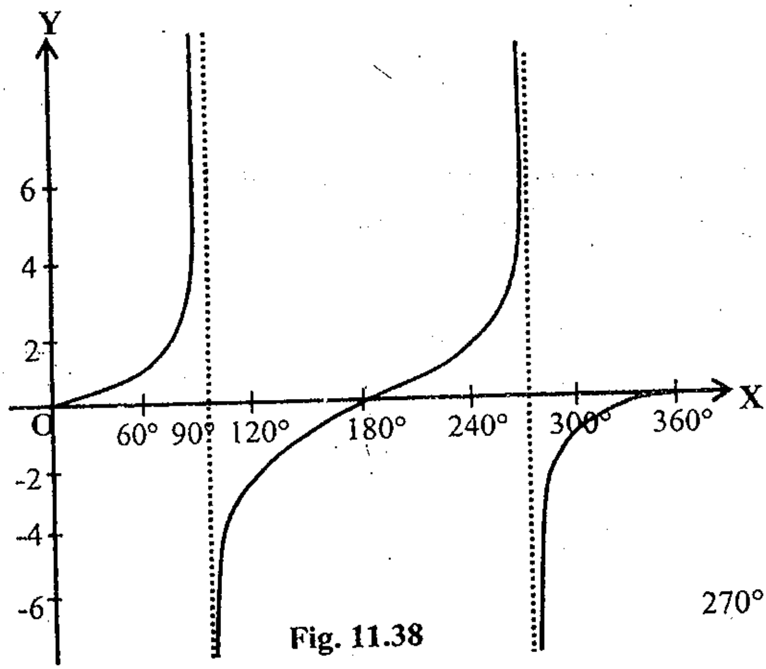


Fig. 11.38

The tangent curve approaches very close to the vertical lines drawn through 90°, 270°, etc as x approaches these values. But the graph never really touches these lines. Such vertical lines are called asymptotes and we say that the graph is nearing them asymptoically.

SUMMARY

1. For any angle θ,

$$
\begin{aligned}
\sin(-\theta) &= -\sin\theta, \\
\cos(-\theta) &= \cos\theta, \\
\tan(-\theta) &= -\tan\theta. \\
\sin(90^\circ - \theta) &= \cos\theta \\
\cos(90^\circ - \theta) &= \sin\theta \\
\tan(90^\circ - \theta) &= \cot\theta \\
\sin(90^\circ + \theta) &= \cos\theta, \\
\cos(90^\circ + \theta) &= -\sin\theta, \\
\tan(90^\circ + \theta) &= -\cot\theta, \\
\sin(180^\circ - \theta) &= \sin\theta \\
\cos(180^\circ - \theta) &= -\cos\theta \\
\tan(180^\circ - \theta) &= -\tan\theta \\
\sin(180^\circ + \theta) &= -\sin\theta \\
\cos(180^\circ + \theta) &= -\cos\theta \\
\tan(180^\circ + \theta) &= \tan\theta
\end{aligned}
$$

2. Important identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\sec\theta = \dfrac{1}{\cos\theta}$, $\quad \sin^2\theta + \cos^2\theta = 1.$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\quad 1 + \tan^2\theta = \sec^2\theta.$

3. $\sin(\alpha \pm \beta) \equiv \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

4. $\cos(\alpha \pm \beta) \equiv \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

5. $\tan(\alpha \pm \beta) \equiv \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$

6. $\sin 2\alpha \equiv 2\sin\alpha\cos\alpha$

7. $\cos 2\alpha \equiv \cos^2\alpha - \sin^2\alpha \equiv 1 - 2\sin^2\alpha \equiv 2\cos^2\alpha - 1$

8. $\tan 2\alpha \equiv \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

9. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

10. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

11. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

12. $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

13. Equations involving multiple angles are usually solved by converting the equation into another equation that contains only one trigonometrical ratio or by using the values from tables.

14. Equations in the form $a\cos\theta + b\sin\theta = c$ are usually solved by converting

$a\cos\theta \pm b\sin\theta$ into $\sqrt{a^2+b^2}\cos(\theta \mp \alpha)$, (or)

$a\sin\theta \pm b\cos\theta$ into $\sqrt{a^2+b^2}\sin(\theta \pm \alpha)$

where $\tan\alpha = \dfrac{b}{a}$, a and b are positive and α is acute.

15. The Law of Cosines and the Law of Sines.

If α, β, γ are the angle opposite to the sides a, b, c respectively, then

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= c^2 + a^2 - 2ca\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}\right\} \quad \text{(Law of Cosines)}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \text{(Law of Sines)}$$

Calculus

12.1 Limits

To approach the subject of calculus, we examize the idea of limits. We want to study the behaviour of a function f(x) when x is near a. In other words, when x is close to a or x approaches a, what happens to f(x)? We write $x \to a$ to represent " x approaches a".

For example, when $x \to 3$, $x = 3.1, 3.01, 3.001, 3.0001, \ldots$ (from the right of 3, $x > 3$) (or) $x = 2.9, 2.99, 2.999, 2.9999, \ldots$ (from the left of 3, $x < 3$).

Notice that when $x \to 3$, x is near 3. $x \simeq 3$, but $x \neq 3$ or $x - 3 \neq 0$.

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$

when $x = 3$, $f(x) = \dfrac{3^2 - 9}{3 - 3} = \dfrac{0}{0}$

The value $\dfrac{0}{0}$ is meaningless and is undefined.

But when $x \to 3$,

$$\begin{aligned} f(x) &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x-3)(x+3)}{(x-3)} \\ &= x + 3 \end{aligned}$$

We were able to cancel $(x - 3)$ because $(x - 3) \neq 0$

Now, we study the following computations.

when $x = 3.1$, $f(x) = 6.1$	when $x = 2.9$, $f(x) = 5.9$
when $x = 3.01$, $f(x) = 6.01$	when $x = 2.99$, $f(x) = 5.99$
when $x = 3.001$, $f(x) = 6.001$	when $x = 2.999$, $f(x) = 5.999$
when $x = 3.0001$, $f(x) = 6.0001$	when $x = 2.9999$, $f(x) = 5.9999$

Notice that when x is close to 3 [from left or right], f(x) gets closer and closer to 6.

" when $x \to 3$, $f(x) \to 6$" is denoted by

$$\lim_{x \to 3} f(x) = 6$$

This is read as " the limit of f(x) is 6 as x tends to 3 ".

Note. $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0.\ \infty$ are called indeterminates.

Example 1.

Find the limit of $f(x) = \dfrac{x^2 - 2x}{x^2 - 4}$ when $x \to 2$.

Solution

$$f(x) = \frac{x^2 - 2x}{x^2 - 4}$$

$$\text{when } x \to 2,\ f(x) = \frac{x(x-2)}{(x+2)(x-2)} = \frac{x}{x+2}$$

$$\therefore \quad \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x}{x+2} = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}$$

Example 2.

Find the limit of

(i) $f(x) = \dfrac{3-4x}{6-7x}$ when $x \to \infty$

(ii) $f(x) = (3x + \frac{1}{x})^2 - (\frac{1}{x} + x)^2$ when $x \to 0$.

Solution

(i) $$f(x) = \frac{3-4x}{6-7x} = \frac{\frac{3}{x} - 4}{\frac{6}{x} - 7}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\frac{3}{x} - 4}{\frac{6}{x} - 7}$$

$$= \frac{0-4}{0-7},\ [\ \frac{3}{x} \to 0 \text{ and } \frac{6}{x} \to 0 \text{ as } x \to \infty]$$

$$= \frac{4}{7}$$

(ii) $f(x) = (3x + \frac{1}{x})^2 - (\frac{1}{x} + x)^2 = 9x^2 + 6 + \frac{1}{x^2} - \frac{1}{x^2} - 2 - x^2$

$= 8x^2 + 4$

$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0}(8x^2 + 4) = 0 + 4 = 4.$

Exercise 12.1

Find the limit of each of the following.

1. $\frac{x^2 - 1}{x + 1}$ as $x \to 1$

2. $\frac{x^2 - 4}{(x+3)(x-4)}$ as $x \to \infty$

3. $\frac{x^2 - x - 12}{x^2 - 11x + 28}$ as $x \to 4$

4. $\frac{(x-4)(x-5)}{(x+1)(x-7)}$ as $x \to \infty$

5. $[(2x - \frac{1}{2x})^2 - (\frac{1}{2x} + 4x)^2]$ as $x \to 0$.

Find the following limits.

6. $\lim_{x \to 1} \frac{x^2 + 1}{x + 2}$

7. $\lim_{x \to 2} (x^2 + 3x + 2)$

8. $\lim_{x \to 2} \frac{2x - x^2}{x^2 - 3x + 2}$

9. $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

10. $\lim_{x \to \infty} \frac{2x^2 - 3x + 1}{x^2 - x + 2}$

11. $\lim_{x \to \infty} \frac{3x^2 - x + 2}{x + 1}$

12. $\lim_{x \to \infty} \frac{x - 2}{2x^2 + 3x - 1}$

13. $\lim_{x \to 0} [(x - \frac{1}{x})^2 - (2x + \frac{1}{x^2})]$

12.2 Derivatives

Let P (x, y) be a point on the graph of the function $y = f(x)$. [Fig.12.1]
If Q $(x + \delta x, y + \delta y)$ is another point on the graph near P, then

$y + \delta y = f(x + \delta x)$

$\therefore \delta y = f(x + \delta x) - f(x)$ where δx represents a small increment in x and δy represents a small increment in y.

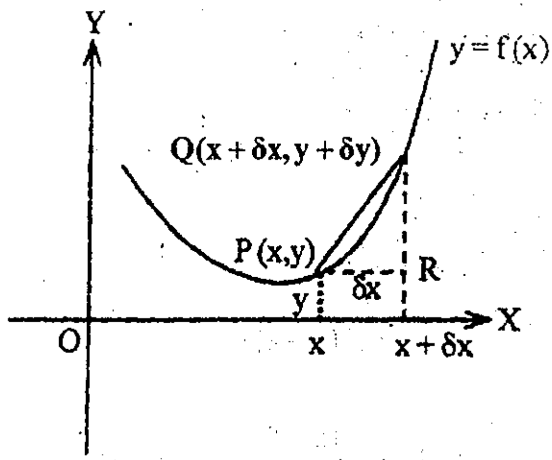


Fig. 12.1

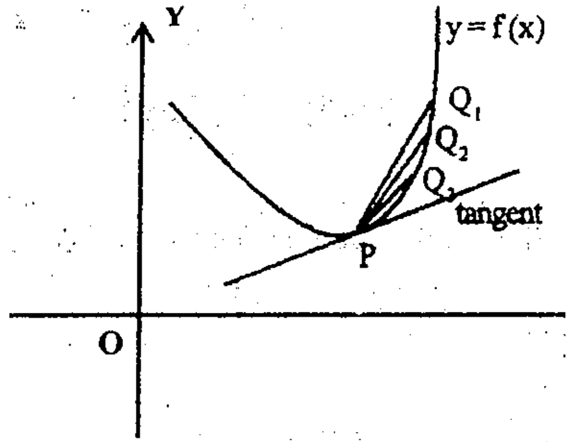


Fig. 12.2

$$\text{Gradient of line PQ} = \frac{QR}{PR} = \frac{\delta y}{\delta x}$$

$$= \frac{f(x+\delta x) - f(x)}{\delta x}$$

As Q approaches P, δx becomes smaller and smaller (i.e. $\delta x \to 0$). Then the gradient of PQ tends to that of the tangent to the curve at P. (Fig. 12.2)

For example, consider the curve $y = x^2$.

Let P (2,4) be a point on the curve.

Consider a point Q_1 on the curve whose x – coordinate is 2.1.

Then y-coordinate of Q_1 is $(2.1)^2 = 4.41$.

$\therefore$ Q_1 (2.1, 4.41).

$$\text{Gardient of } PQ_1 = \frac{4.41-4}{2.1-2} = 4.1.$$

Now consider another point Q_2 which is closer to P than Q_1 whose x-coordinate is 2.01. Then the y-coordinate of Q_2 is $(2.01)^2 = 4.0401$.

$\therefore$ Q_2 (2.01, 4.0401).

$$\text{Gardient of } PQ_2 = \frac{4.0401-4}{2.01-2} = 4.01.$$

The table below shows the values of the gradient when the points Qs are closer and closer to P taken.

x-coordinate of Q	y-coordinate of Q	gradient of PQ
2.1	4.41	4.1
2.01	4.0401	4.01
2.001	4.004001	4.001
2.0001	4.00040001	4.0001

The results from the table show that the gradient of PQ tends to 4 as Q approaches P.

Since the gradient of PQ tends to the gradient of the tangent to the curve at P as Q approaches P,

the gradient of tangent at P $= \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

The limit $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ is called **the derivative of y = f(x) with respect to x** (or) **the rate of change of y = f(x) with respect to x** and is denoted by

$$\frac{dy}{dx} \text{ (or) } \frac{df(x)}{dx} \text{ (or) } y' \text{ (or) } f'(x).$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The entire procedure leading to the derivative is called the **differentiation from the first principles.**

The derivative of $y = f(x)$, at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad \text{where h is a small increment in a.}$$

For the curve $y = f(x)$, the gradient of the tangent l_1 at the point (x_1, y_1) is the value of $\dfrac{dy}{dx}$ at $x = x_1$.

Hence, the **equation of the tangent** at (x_1, y_1) is

$$y - y_1 = \frac{dy}{dx}(x - x_1).$$

The line l_2 which is perpendicular to the tangent l_1 at (x_1, y_1) is called the **normal** to the curve at (x_1, y_1).

Hence its gradient is the value of $-\dfrac{1}{\frac{dy}{dx}}$ at $x = x_1$ and the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{\frac{dy}{dx}}(x - x_1).$$

Example 1.

Differentiate $x^2 + 3x + 6$ with respective to x from the first principles.

Solution

Let $y = x^2 + 3x + 6$

$$y + \delta y = (x + \delta x)^2 + 3(x + \delta x) + 6$$

$$= x^2 + 2x\,.\,\delta x + (\delta x)^2 + 3x + 3\,.\,\delta x + 6$$

$$\therefore \delta y = 2x\,.\,\delta x + (\delta x)^2 + 3.\,\delta x$$

$$\frac{\delta y}{\delta x} = \frac{2x.\delta x + (\delta x)^2 + 3.\delta x}{\delta x} = 2x + \delta x + 3$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x + 3) = 2x + 3.$$

Example 2.

Differentiate $y = \dfrac{1}{x}$ with respect to x from the first principles.

Solution

$$y = \frac{1}{x}$$

$$y + \delta y = \frac{1}{x + \delta x}$$

$$\delta y = \frac{1}{x + \delta x} - \frac{1}{x} = \frac{x - x - \delta x}{(x + \delta x).x} = \frac{-\delta x}{(x + \delta x).x}$$

$$\frac{\delta y}{\delta x} = \frac{-1}{(x + \delta x).x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{-1}{(x+\delta x).x} = \frac{-1}{x^2}$$

Example 3.

Differentiate $f(x) = \sqrt{x}$ with respect to x from the first principles.

Solution

$$f(x) = \sqrt{x}$$

$$f(x+\delta x) = \sqrt{x+\delta x}$$

$$f(x+\delta x) - f(x) = \sqrt{x+\delta x} - \sqrt{x}$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x} \times \frac{\sqrt{x+\delta x} + \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}$$

$$= \lim_{\delta x \to 0} \frac{x+\delta x - x}{\delta x[\sqrt{x+\delta x} + \sqrt{x}]} = \lim_{\delta x \to 0} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Example 4.

Differentiate $f(x) = x^2 + 5$ with respect to x at x = 3 from the first principles.

Solution

$$f(x) = x^2 + 5$$

$$f(3) = 3^2 + 5 = 9 + 5 = 14$$

$$f(3+h) = (3+h)^2 + 5 = 9 + 6h + h^2 + 5 = 14 + 6h + h^2$$

$$f(3+h) - f(3) = 6h + h^2 = h(6+h)$$

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h \to 0} (6+h) = 6$$

Exercise 12.2

Find the derivatives of the following functions from the first principles.

(1) x^3, (2) $\sqrt[3]{x}$, (3) $\frac{1}{x^2}$, (4) $\frac{1}{\sqrt{x}}$, (5) $1 - 2x^2$ at $x = 2$

12.3 Some Particular Derived Functions

(i) The derivative of a constant function.

Let $f(x) = C$, where C is a constant.

Then $f(x + \delta x) = C$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{C - C}{\delta x} = 0$$

$\therefore$ the derivative of a constant function is zero.

i.e., $\frac{dC}{dx} = 0.$

(ii) The derivative of x^n where n is a positive integer.

Let $f(x) = x^n$, n is a positive integer.

$f(x + \delta x) = (x + \delta x)^n$

$= x^n + n.x^{n-1} . \delta x + \frac{n(n-1)}{2} . x^{n-2} .(\delta x)^2 + \ldots + (\delta x)^n.$

(Binomial theorem)

$= x^n + \delta x [n . x^{n-1} + \frac{n(n-1)}{2} . x^{n-2} .(\delta x) + \ldots]$

$\therefore f(x + \delta x) - f(x) = \delta x [n. x^{n-1} + \frac{n(n-1)}{2} . x^{n-2} . \delta x + \ldots]$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta x[n.x^{n-1} + \frac{n(n-1)}{2}.x^{n-2}.\delta x + \ldots 1]}{\delta x}$$

$$= \lim_{\delta x \to 0} [n.x^{n-1} + \frac{n(n-1)}{2}.x^{n-2}.\delta x + \ldots] = n. x^{n-1}$$

$\therefore \dfrac{d}{dx}(x^n) = n . x^{n-1}$, where n is a positive integer.

It is true for n is a negative integer or other rational number.
In general,

$$\frac{d}{dx}(x^n) = n . x^{n-1} \text{ where n is an integer or a rational number.}$$

Again we shall state without proof that

(i) $\dfrac{d}{dx}[u(x) \pm v(x)] = \dfrac{d}{dx}u(x) \pm \dfrac{d}{dx}v(x)$

(ii) $\dfrac{d}{dx}[C.u(x)] = C.\dfrac{d}{dx}u(x)$, where u(x) and v(x) are functions of x and C is a constant.

Example 1.

Find $\dfrac{dy}{dx}$.

(i) $y = 3x^2$, (ii) $y = \dfrac{1}{x}$, (iii) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, (iv) $y = x^3 + 2x^2 - 3x - 6$.

Solution

(i) $y = 3x^2$.

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2) = 3 . \frac{dx^2}{dx} = 3\ (2x) = 6x$$

(ii) $y = \dfrac{1}{x} = x^{-1}$

$$\frac{dy}{dx} = (-1).x^{-2} = \frac{-1}{x^2}$$

(iii) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} = \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) = \frac{d}{dx}x^{\frac{1}{2}} + \frac{d}{dx}x^{-\frac{1}{2}}.$$

$$= \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

(iv) $y = x^3 + 2x^2 - 3x - 6$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 2x^2 - 3x - 6)$$

$$= \frac{dx^3}{dx} + 2\frac{dx^2}{dx} - 3\frac{dx}{dx} - \frac{d6}{dx}$$

$$= 3x^2 + 2(2x) - 3(1) - 0 \quad = \quad 3x^2 + 4x - 3$$

Example 2.

Find the derivatives of the following with respect to x.

(i) $(x + 1)(x + 2)$, (ii) $(3x - 2)^2$, (iii) $\frac{2x^3 - 3x^2}{4\sqrt{x}}$

Solution

(i) $\frac{d}{dx}[(x + 1)(x + 2)] = \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$

(ii) $\frac{d}{dx}(3x - 2)^2 = \frac{d}{dx}(9x^2 - 12x + 4) = 9(2x) - 12$

$= 18x - 12$

(iii) $\frac{d}{dx}\left[\frac{2x^3 - 3x^2}{4\sqrt{x}}\right] = \frac{d}{dx}\left(\frac{2x^3}{4\sqrt{x}}\right) - \frac{d}{dx}\left(\frac{3x^2}{4\sqrt{x}}\right)$

$$= \frac{1}{2}\frac{d}{dx}(x^{5/2}) - \frac{3}{4}\frac{d}{dx}x^{3/2}$$

$$= \frac{1}{2}\cdot\frac{5}{2}\cdot x^{3/2} - \frac{3}{4}\cdot\frac{3}{2}\cdot x^{1/2}$$

$$= \frac{5}{4}x^{3/2} - \frac{9}{8}x^{1/2}$$

Example 3.

Given $f(x) = (x^2 - 3)^2$, find $f'(x)$ and $f'(-1)$.

Solution

$f(x) = (x^2 - 3)^2 = x^4 - 6x^2 + 9$

$f'(x) = 4x^3 - 6(2x) = 4x^3 - 12x$

$f'(-1) = 4(-1)^3 - 12(-1) = -4 + 12 = 8$

Example 4.

Given that $A = 2r^2 - 4r + 5$, find the rate of change of A with respect to r when r = 3.

Solution

$A = 2r^2 - 4r + 5$

$\frac{dA}{dr} = 4r - 4$

when r = 3, $\frac{dA}{dr} = 4(3) - 4 = 12 - 4 = 8$.

Example 5.

Find the gradient of the curve $y = 3x^2 - 4x + 3$ at the point where x = 2.

Solution

Curve: $y = 3x^2 - 4x + 3$

$\frac{dy}{dx} = 3(2x) - 4 = 6x - 4$

when x = 2, $\frac{dy}{dx} = 6(2) - 4 = 12 - 4 = 8$.

$\therefore$ The gradient of the curve at the point w.. re x = 2 is 8.

Example 6.

Find the equations of the tangent and the normal line to the curve $y = x^2 - 3x + 2$ at the point where x = 3.

Solution

Curve: $y = x^2 - 3x + 2$

when $x = 3, y = 3^2 - 3(3) + 2 = 9 - 9 + 2 = 2$.

$\therefore$ point A (3,2).

$\frac{dy}{dx} = 2x - 3$

when x = 3, $\frac{dy}{dx} = 2(3) - 3 = 3$.

$\therefore$ The gradient of the tangent line to the curve at A (3,2) is 3.

$\therefore$ Equation of the tangent line to the curve at A (3,2) is

$y - y_1 = m(x - x_1)$, [Formula for equation of the line withgradient m and through the point (x_1, y_1)]

$\therefore y - 2 = 3(x - 3)$

$\therefore 3x - y = 7$

Equation of the normal line at A (3,2) is

$$y - y_1 = \frac{-1}{m}(x - x_1), \text{ [tangent line } \perp \text{ normal line]}$$

$$y - 2 = \frac{-1}{3}(x - 3)$$

$$x + 3y = 9.$$

Example 7.

When a marble is moving in a groove, the distance s cm from one end at time

t sec is given by $s = 5t - t^2$.

(a) Find the speed of the marble at t = 2 sec.

(b) Find t when the speed of the marble is zero.

Solution

$$s = 5t - t^2$$

$$\frac{ds}{dt} = 5 - 2t$$

(a) when t = 2 sec , the speed $\frac{ds}{dt} = 5 - 2(2) = 5 - 4 = 1$ cm / sec

(b) when the speed is zero,

$$\frac{ds}{dt} = 0$$

$$5 - 2t = 0$$

$$t = 2.5 \text{ sec.}$$

Exercise 12.3

1. Differentiate the following with respect to x.

(i) $4x^3$ (ii) $\frac{4}{x^3}$ (iii) $\frac{2}{\sqrt[3]{x}}$ (iv) $x^3 + \frac{1}{\sqrt{x}}$

(v) $x^2 - \frac{1}{x} - \frac{3}{x^2}$ (vi) $\frac{3x^2 - 4\sqrt{x} + 1}{x}$ (vii) $(3x + 1)(2 - x)$

2. Find $\frac{dy}{dx}$.

(i) $y = x(1-x^2)^2$ (ii) $y = 8x^{\frac{3}{4}} - \frac{6}{x^{2/3}}$ (iii) $y = (\sqrt{x} + \frac{1}{\sqrt{x}})^2$

(iv) $y = \frac{(1-x)(3x+2)}{\sqrt{x}}$ (v) $y = (x - 1 + \frac{1}{x})(x - 1 - \frac{1}{x})$.

3. Given tha $(x) = 4 . x^{1/2}$, find $f'(x)$ and then $f'(1)$, $f'(4)$, $f'(\frac{1}{9})$

4. Calculate the rate of change of the function $f : x \longmapsto \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$ at $x = 8$.

5. Given $f(x) = (x + \frac{1}{x})(x - \frac{1}{x})$. Show that $f'(x) = \frac{2x^4 + 2}{x^3}$.

6. Given that $V = \frac{4}{3}r^3 - \frac{3}{4}r^2 + r - 5$, find the rate of change of V with respect to r when $r = 2$.

7. Given that the gradient of the curve $y = x^2 + ax + b$ at the point $(2, -1)$ is 1. Find the values of a and b.

8. Find the equation of the tangent to the curve $y = x + 5x - 2$ at the point on the curve where this curve cuts the line $x = 4$.

9. Find the equations of the tangent and normal lines to the curve $y = x^2 - 5x + 6$ at the points where this curve cuts the x-axis.

10. Find the equation of the normal line to the curve $y = x^2 - 3x + 2$ which has gradient of $\frac{1}{2}$.

12.4 Chain rule, Product rule and Quotient rule.

(I) Chain rule

Suppose that y is a function of u and u is a function of x.

i.e. $y = f(u)$ and $u = g(x)$.

If δx, δy and δu are the small increments in x, y and u respectively, then

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

when $\delta x \to 0$, and $\delta u \to 0$, then

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This technique of differentiation is known as " the function of a function method " or " **the chain rule** ".

Consider $y = u^n$ and $u = u(x)$. Then by using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{du^n}{du} \cdot \frac{du}{dx}$$

$$= n \cdot u^{n-1} \cdot \frac{du}{dx}$$

e get the formula

$$\frac{d}{dx}[u(x)]^n = n\,[u(x)]^{n-1} \cdot \frac{d}{dx} u(x)$$, where n is an integer or a rational number.

Example 1.

Differentiate the following with respect to x.

(i) $(2x^2 + 3x)^{10}$, (ii) $\frac{1}{3-2x}$, (iii) $\sqrt{9-x^2}$, (iv) $(x^2 + \frac{3}{x})^5$.

Solution

(i) $\frac{d}{dx}(2x^2 + 3x)^{10} = 10(2x^2 + 3x)^9 \cdot \frac{d}{dx}(2x^2 + 3x)$

$= 10(2x^2 + 3x)^9 \cdot (4x + 3)$

(ii) $\frac{d}{dx}(\frac{1}{3-2x}) = \frac{d}{dx}(3-2x)^{-1}$

$= (-1)(3-2x)^{-2} \cdot \frac{d}{dx}(3-2x)$

$= (-1)(3-2x)^{-2} \cdot (-2) = \frac{2}{(3-2x)^2}$.

(iii) $\frac{d}{dx}\sqrt{9-x^2} = \frac{d}{dx}(9-x^2)^{\frac{1}{2}}$

$= \frac{1}{2}(9-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(9-x^2)$

$= \frac{1}{2}(9-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{9-x^2}}$

(iv) $\frac{d}{dx}(x^2 + \frac{3}{x})^5 = 5(x^2 + \frac{3}{x})^4 \cdot \frac{d}{dx}(x^2 + 3x^{-1})$

$$= 5(x^2 + \frac{3}{x})^4 . (2x + 3(-1)x^{-2})$$

$$= 5(x^2 + \frac{3}{x})^4 . (2x - \frac{3}{x^2})$$

(2) Product rule

Let y = uv where u and v are functions of :

Let δx be small increment in x and δu, δv, δy be the corresponding increments in u, v, y respectively .

Then $\quad y + \delta y = (u + \delta u)(v + \delta v)$

$$= uv + u . \delta v + v . \delta u + \delta u . \delta v$$

$$\delta y = u . \delta v + v . \delta u + \delta u . \delta v$$

$$\frac{\delta y}{\delta x} = u . \frac{\delta v}{\delta x} + v . \frac{\delta u}{\delta x} + \frac{\delta u . \delta v}{\delta x}$$

when $\delta x \to 0$, then $\delta u \to 0$ and $\delta v \to 0$.

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}, \qquad \lim_{\delta x \to 0} \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx} \text{ and}$$

$$\lim_{\delta x \to 0} \frac{\delta u . \delta v}{\delta x} = \lim_{\delta x \to 0} \delta u \times \lim_{\delta x \to 0} \frac{\delta v}{\delta x} \text{ (or) } \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \times \lim_{\delta x \to 0} \delta v$$

$$= 0 \times \frac{dv}{dx} \text{ (or) } \frac{du}{dx} \times 0$$

$$= 0$$

Hence $\quad \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\therefore \frac{d}{dx}[u . v] = u \frac{dv}{dx} + v \frac{du}{dx}$$

This is called **the product rule.**

Example 2.

Differentiate $\sqrt{x+7} . (x^2 + 2)^7$ with respect to x.

Solution

$$\frac{d}{dx}[\sqrt{x+7} . (x^2+2)^7] = \sqrt{x+7} \frac{d}{dx}(x^2+2)^7 + (x^2+2)^7 . \frac{d}{dx}(x+7)^{1/2}$$

$$= \sqrt{x+7}\,.\,7(x^2+2)^6\,.\,2x + (x^2+2)^7\,.\,\frac{1}{2}(x+7)^{-\frac{1}{2}}$$

(3) Quotient rule

Let $y = \dfrac{u}{v}$ where u and v are both functions of x and $v(x) \neq 0$.

$$y + \delta y = \frac{u+\delta u}{v+\delta v}$$

$$\therefore \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v}$$

$$= \frac{(u+\delta u)v - (v+\delta v)u}{(v+\delta v)v} = \frac{v.\delta u - u.\delta v}{(v+\delta v)v}$$

$$\frac{\delta y}{\delta x} = \frac{v.\frac{\delta u}{\delta x} - u.\frac{\delta v}{\delta x}}{(v+\delta v)v}$$

when $\delta x \to 0$, then $\delta u \to 0$ and $\delta v \to 0$.

$\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \dfrac{dy}{dx}$, $\lim\limits_{\delta x \to 0} \dfrac{\delta u}{\delta x} = \dfrac{du}{dx}$, $\lim\limits_{\delta x \to 0} \dfrac{\delta v}{\delta x} = \dfrac{dv}{dx}$. Then

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

$$\therefore \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}, \ (v \neq 0)$$

This is the **quotient rule.**

Example 3.

Differentiate $\dfrac{x^2}{\sqrt{x^2+1}}$ with respect to x.

Solution

$$\frac{d}{dx}\left(\frac{x^2}{\sqrt{x^2+1}}\right) = \frac{\sqrt{x^2+1}.\frac{d}{dx}x^2 - x^2.\frac{d}{dx}(x^2+1)^{\frac{1}{2}}}{x^2+1} \quad \text{(by quotient rule)}$$

$$= \frac{\sqrt{x^2+1}.2x - x^2.\frac{1}{2}(x^2+1)^{-1/2}.2x}{(x^2+1)}$$

$$= \frac{(x^2+1).2x - x^3}{(x^2+1)^{3/2}} \quad = \frac{x^3+2x}{(x^2+1)^{3/2}}$$

If we use the product rule,

$$\frac{d}{dx}\left(\frac{x^2}{\sqrt{x^2+1}}\right) = \frac{d}{dx}\, x^2 \, . \, (x^2+1)^{-1/2}$$

$$= x^2 . \frac{d}{dx}(x^2+1)^{-1/2} + (x^2+1)^{-1/2} . \frac{d}{dx}\, x^2$$

$$= x^2 . \left(-\frac{1}{2}\right)(x^2+1)^{-3/2}.2x + (x^2+1)^{-1/2}.2x$$

$$= \frac{-x^3}{(x^2+1)^{3/2}} + \frac{2x}{(x^2+1)^{1/2}} = \frac{-x^3+2x(x^2+1)}{(x^2+1)^{3/2}}$$

$$= \frac{x^3+2x}{(x^2+1)^{3/2}}$$

Higher order derivatives

When a function $y = f(x)$ is differentiated with respect to x, the derivative $\frac{dy}{dx}$ is also a function of x.This function can be differentiated again with respect to x, giving $\frac{d}{dx}\left(\frac{dy}{dx}\right)$. This is called the **second derivative** of $y = f(x)$ with respect to x and is written by

$\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}$ f(x) or y'' (or) $f''(x)$.

This function $\frac{d^2y}{dx^2}$ is also a function of x.

$\frac{d^2y}{dx^2}$ is further differentiated with respect to x.

The **third derivative** is written by $\frac{d^3y}{dx^3}$ or $\frac{d^3}{dx^3}$ f(x) or y''' (or) $f'''(x)$.

Similarly , we can state the fourth , fifth , ... etc derivatives.

These derivatives $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$ are the higher order derivatives of $y = f(x)$ with respect to x.

Example 4.

Let $y = 5x^3 + 7x^2 + 6$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$.

Solution

$$y = 5x^3 + 7x^2 + 6$$
$$\frac{dy}{dx} = 5\,(3x^2) + 7\,(2x) \quad = 15x^2 + 14x$$
$$\frac{d^2y}{dx^2} = 15\,(2x) + 14 \quad = 30x + 14$$
$$\frac{d^3y}{dx^3} = 30.$$

Example 5.

If $f(x) = x^3 - 2x^2 + 3x + 1$, find $f'(1)$ and $f''(1)$.

Solution

$$f(x) = x^3 - 2x^2 + 3x + 1$$
$$f'(x) = 3x^2 - 2\,(2x) + 3 \quad = 3x^2 - 4x + 3$$
$$f''(x) = 3\,(2x) - 4 \quad = 6x - 4$$
$$\therefore f'(1) = 3\,(1)^2 - 4\,(1) + 3 = 2$$
$$\therefore f''(1) = 6\,(1) - 4 \quad = 2$$

Example 6.

If $y = 3x^2 + 4x$, prove that $x^2 . \frac{d^2y}{dx^2} - 2x . \frac{dy}{dx} + 2y = 0$

Solution

$$y = 3x^2 + 4x$$
$$\frac{dy}{dx} = 3\,(2x) + 4 \quad = 6x + 4$$
$$\frac{d^2y}{dx^2} = 6$$
$$x^2 . \frac{d^2y}{dx^2} - 2x . \frac{dy}{dx} + 2y = x^2(6) - 2x\,(6x + 4) + 2\,(3x^2 + 4x)$$
$$= 6x^2 - 12\,x^2 - 8x + 6x^2 + 8x \quad = 0$$

Exercise 12.4

1. Differentiate the following with respect to x.

   (i) $(2x^2+3)^4(x^2-3x)^5$, (ii) $(3+x^2)\sqrt{3-x^2}$, (iii) $\dfrac{\sqrt{x+3}}{x+1}$

   (iv) $\dfrac{3x-5}{2x^2+7}$, (v) $\dfrac{2x-7}{\sqrt{x+7}}$, (vi) $\sqrt{\dfrac{x^2+1}{x^2-1}}$

2. Calculate the gradient of the curve $y=\dfrac{3x^2-8}{5-2x}$ at the point (2,4).

3. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following functions.

   (i) $y=\dfrac{x}{x-1}$, (ii) $y=x\sqrt{x+2}$, (iii) $y=\dfrac{x+1}{x^2}$,

   (iv) $y=(3x^2-2x+1)^2$, (v) $y=(3x+2)^{10}$

4. If $y=\dfrac{2x^2+3}{x}$, prove that $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=y$.

5. If $y=x^2+2x+3$, show that $\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{d^2y}{dx^2}\right)^3=4y$.

12.5 Differentiation of Implicit Functions

All the differentiation carried out so far has involved equations of the form $y = f(x)$.

Now consider the curve whose equation is $y+xy+y^2=2$.

This equation is not easily transposed to the form $y=f(x)$ and we say that $y = f(x)$ is implied by the equation $y+xy+y^2=2$.

i.e. f(x) is an **implicit function.**

An implicit function can sometimes be changed into an explicit function (expressing y in terms of x). However , it is sometimes difficult and unnecessary to do so.

Example 1.

Find $\dfrac{dy}{dx}$ if $x^2-xy^2-y^3=2$

Solution

$x^2-xy^2-y^3=2$

Differentiate with respect to x.

$$2x - [\, x.\frac{dy^2}{dx} + y^2.\frac{dx}{dx}] - 3y^2.\frac{dy}{dx} = 0$$

$$2x - x.2y\frac{dy}{dx} - y^2 - 3y^2.\frac{dy}{dx} = 0$$

$$(2xy + 3y^2)\frac{dy}{dx} = 2x - y^2$$

$$\frac{dy}{dx} = \frac{2x - y^2}{2xy + 3y^2}$$

Example 2.

Find the equation of the tangent line to the curve $3x^2 + 2y^2 = 2xy + 23$ at the point (3,2).

Solution

Curve: $3x^2 + 2y^2 = 2xy + 23$

Differentiate with respect to x,

$$6x + 4y\frac{dy}{dx} = 2[\, x.\frac{dy}{dx} + y\,]$$

$$(4y - 2x)\frac{dy}{dx} = 2y - 6x$$

$$\frac{dy}{dx} = \frac{y - 3x}{2y - x}$$

$\therefore$ The gradient of tangent to the curve at (3,2) is

$$m = \frac{2 - 3(3)}{2(2) - 3} = -7$$

$\therefore$ Equation of the tangent line is

$$y - 2 = -7\ (x - 3)$$

$$\therefore\ 7x + y - 23 = 0$$

Exercise 12.5

1. Find $\frac{dy}{dx}$.

(i) $xy = 5$,

(ii) $x\,(x + y) = y^2$,

(iii) $x^3 - 4\,xy + y^2 = 14$,

(iv) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$

2. Show that the equation of the tangent to the curve $x^2 + xy + y = 0$ at the point (a,b) is $x(2a+b) + y(a+1) + b = 0$.

3. Find the coordinates of the points on the curve $x^2 - y^2 = 3xy - 39$ at which the tangents are (i) parallel (ii) perpendicular to the line $x + y = 1$.

12.6 Differentiation of Trigonometric Functions

Before we study the differentiation of trigonometric functions, we first evaluate an important limit, $\lim_{x \to 0} \frac{\sin x}{x}$.

Consider the unit circle in figure 12.3 with OA = OB = 1 (radius) and $\angle AOB = x$ radian. Obviously, BD < arcAB < AC. In right ΔOBD, BD = OB sinx = sinx, ($\because$ OB = 1)

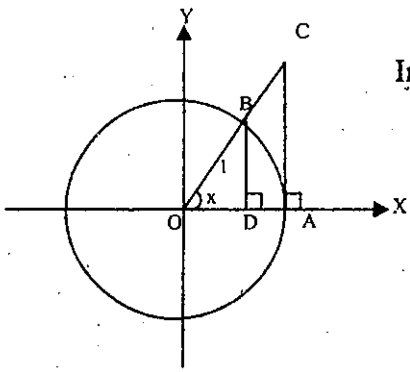


Fig. 12.3

In right ΔOAC, AC = OA tanx = tanx, ($\because$ OA = 1)

Length of arc AB = OB (x) = x

$\therefore \sin x < x < \tan x$

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x}, \ (\sin x > 0)$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

When $x \to 0$, $\cos x = 1$,

Since $\lim_{x \to 0} 1 = 1$ and $\lim_{x \to 0} \cos x = 1$,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Note. In this chapter, all angles are measured in radian unless otherwise stated.

Derivatiye of sin x

$$\text{Let} \quad y = \sin x$$
$$y + \delta y = \sin (x + \delta x)$$
$$\therefore \quad \delta y = \sin (x + \delta x) - \sin x$$
$$= 2.\cos \left(x + \frac{\delta x}{2}\right).\sin \frac{\delta x}{2}, \quad \text{[using the formula}$$
$$\sin \alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}]$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2.\cos(x + \frac{\delta x}{2}).\sin\frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right). \lim_{\delta x \to 0} \frac{\sin(\frac{\delta x}{2})}{(\frac{\delta x}{2})}$$

$$= \cos x \times 1, \qquad [\text{ when } \delta x \to 0, \ \frac{\delta x}{2} \to 0]$$

$$= \cos x$$

$$\therefore \frac{d}{dx} \sin x = \cos x$$

In general, $\frac{d}{dx} \sin u(x) = \cos u(x) . \frac{d}{dx} u(x)$

Derivative of cos x

$$\text{Since } \cos x = \sin\left(\frac{\pi}{2} + x\right)$$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin\left(\frac{\pi}{2} + x\right) = \cos\left(\frac{\pi}{2} + x\right) . \frac{d}{dx}\left(\frac{\pi}{2} + x\right)$$

$$= -\sin x \times 1, \qquad [\because \cos\left(\frac{\pi}{2} + x\right) = -\sin x\,]$$

$$\therefore \frac{d}{dx} \cos x = -\sin x$$

In general, $\frac{d}{dx} \cos u(x) = -\sin u(x) \ \frac{d}{dx} u(x)$.

Derivative of tan x

$$\text{Since } \tan x = \frac{\sin x}{\cos x}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x.\frac{d}{dx}\sin x - \sin x.\frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x} \quad [\text{ quotient formula }]$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \ [\because \cos^2 x + \sin^2 x = 1 \ ]$$

$$= \sec^2 x \quad . [\because \sec x = \frac{1}{\cos x}]$$

$$\therefore \frac{d}{dx}\tan x = \sec^2 x$$

In general, $\frac{d}{dx}\tan u(x) = \sec^2 u(x) . \frac{d}{dx}u(x)$.

Similarly, we can easily find the formulas for the derivatives of cot x, sec x and cosec x .

$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$, $\quad \frac{d}{dx}\cot u(x) = -\operatorname{cosec}^2 u(x) . \frac{d}{dx}u(x)$.

$\frac{d}{dx}\sec x = \sec x . \tan x$, $\quad \frac{d}{dx}\sec u(x) = \sec u(x) . \tan u(x) . \frac{d}{dx}u(x)$.

$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x . \cot x$,

$\frac{d}{dx}\operatorname{cosec} u(x) = -\operatorname{cosec} u(x) . \cot u(x) . \frac{d}{dx}u(x)$.

Formulas for derivatives of trigonometric functions

1	$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u . \frac{du}{dx}$.
2	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u . \frac{du}{dx}$.
3	$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u . \frac{du}{dx}$.

4	$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u.\frac{du}{dx}.$
5	$\frac{d}{dx}\sec x = \sec x.\tan x$	$\frac{d}{dx}\sec u = \sec u.\tan u.\frac{du}{dx}.$
6	$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x.\cot x$	$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u.\cot u.\frac{du}{dx}$

Example 1.

Differentiate the following with respect to x.

(i) $\sin 5x$, (ii) $\cos(7x^2-2)$, (iii) $\tan(6x+7)$

(iv) $5\sec(3x+1)$, (v) $\frac{\cot(1-2x)}{3}$, (vi) $-2\operatorname{cosec} 3x$.

Solution

(i) $\frac{d}{dx}\sin 5x = \cos 5x.\frac{d}{dx}5x = \cos 5x.5 = 5.\cos 5x.$

(ii) $\frac{d}{dx}\cos(7x^2-2) = -\sin(7x^2-2).\frac{d}{dx}(7x^2-2) = -\sin(7x^2-2).(14x)$

(iii) $\frac{d}{dx}\tan(6x+7) = \sec^2(6x+7).\frac{d}{dx}(6x+7) = \sec^2(6x+7).6$

(iv) $\frac{d}{dx}5\sec(3x+1) = 5.\sec(3x+1).\tan(3x+1).\frac{d}{dx}(3x+1)$

$= 5.\sec(3x+1).\tan(3x+1).3$

$= 15.\sec(3x+1).\tan(3x+1)$

(v) $\frac{d}{dx}\frac{\cot(1-2x)}{3} = \frac{-1}{3}.\operatorname{cosec}^2(1-2x).\frac{d}{dx}(1-2x)$

$= \frac{-1}{3}.\operatorname{cosec}^2(1-2x).(-2) = \frac{2}{3}\operatorname{cosec}^2(1-2x).$

(vi) $\frac{d}{dx}[-2\operatorname{cosec} 3x] = (-2).(-\operatorname{cosec} 3x.\cot 3x).\frac{d}{dx}(3x)$

$= 2\operatorname{cosec} 3x.\cot 3x.3 = 6\operatorname{cosec} 3x.\cot 3x.$

Example 2.

Find $\frac{dy}{dx}$.

(i) $y = \sin^2 x$, (ii) $y = \cos\sqrt{x}$, (iii) $y = \tan^2(x^2)$

(iv) $y = \sin 2x - x\cos x$, (v) $y = \sin x \,.\, \cos^2 x$, (vi) $y = \frac{x}{\tan x}$

(vii) $y = \sqrt{x + \sin x}$

Solution

(i) $y = \sin^2 x$

$$\frac{dy}{dx} = 2\sin x \,.\, \cos x$$

(ii) $y = \cos\sqrt{x} = \cos x^{1/2}$

$$\frac{dy}{dx} = -\sin\sqrt{x} \,.\, \frac{1}{2}x^{-1/2}$$

(iii) $y = \tan^2(x^2)$

$$\frac{dy}{dx} = 2.\tan(x^2) \,.\, \sec^2(x^2).2x$$

(iv) $y = \sin 2x - x\cos x$

$$\frac{dy}{dx} = (\cos 2x)\,2 - [x\,.(-\sin x) + \cos x\,]$$

$$= 2\cos 2x + x\sin x - \cos x$$

(v) $y = \sin x \,.\, \cos^2 x$

$$\frac{dy}{dx} = \sin x \,.\, \frac{d}{dx}(\cos^2 x) + \cos^2 x.\ \frac{d}{dx}(\sin x)$$

$= \sin x\,.2\cos x.\,(-\sin x) + \cos^2 x\,.\cos x = -2\sin^2 x.\cos x + \cos^3 x$

(vi) $y = \frac{x}{\tan x}$

$$\frac{dy}{dx} = \frac{\tan x - x.\sec^2 x}{\tan^2 x}$$

(vii) $y = \sqrt{x+\sin x} = (x+\sin x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(x+\sin x)^{-1/2} . (1+\cos x)$$

Example 3.

Given that $x + \sin y = \cos(xy)$, find $\frac{dy}{dx}$.

Solution $x + \sin y = \cos(xy)$

Differentiate with respect to x,

$$1 + \cos y.\frac{dy}{dx} = -\sin(xy).\frac{d}{dx}(xy)$$

$$1 + \cos y.\frac{dy}{dx} = -\sin(xy).[x.\frac{dy}{dx} + y]$$

$$(\cos y + x\sin(xy))\frac{dy}{dx} = -(1 + y.\sin(xy))$$

$$\frac{dy}{dx} = \frac{-(1+y.\sin(xy))}{(\cos y + x\sin(xy))}$$

Example 4.

Given that $y = x\sin x$, find $\frac{d^2y}{dx^2}$.

Solution

$$y = x.\sin x$$

$$\frac{dy}{dx} = x.\cos x + \sin x$$

$$\frac{d^2y}{dx^2} = x(-\sin x) + \cos x + \cos x = 2\cos x - x\sin x$$

Exercise 12.6

1. Differentiate the following functions with respect to x.

(i) $\sin(2x+3)$, (ii) $\cos\frac{3}{x}$, (iii) $x^3\cos 2x$, (iv) $\cos 7x + \sin 3x$

(v) $\sin x.\cos 2x$, (vi) $\cos^2(5x)$, (vii) $\tan^3\sqrt{x}$, (viii) $\sin(\cos x)$

(ix) $\frac{\sin x}{1+\tan x}$, (x) $\sqrt{\sin x + \cos x}$

2. Find $\frac{dy}{dx}$.

(i) $y = \sin(1-x^2)$, (ii) $y = 2\pi x + 2\cos\pi x$.

(iii) $y = \sin^2 x.\cos 3x$, (iv) $y = x^2\sin(\frac{1}{x})$.

(v) $3x^2 + 2\sin y = y^2$, (vi) $\sin x.\cos y = 2y$.

3. Given that $y = \cos^2 x$, prove that $\frac{d^2y}{dx^2} + 4y = 2$.

4. Given that $y = \frac{1}{3}\cos^3 x - \cos x$, prove that $\frac{dy}{dx} = \sin^3 x$.

12.7 Application of Differentiations

Sign of the derivative

If $y = f(x)$, then $\frac{dy}{dx}$ or $f'(x)$ gives the rate of change of y with respect to x .

When $\frac{dy}{dx} > 0$, it means that y **increases** as x **increases.**

When $\frac{dy}{dx} < 0$, it means that y **decreases** as x **increases.**

Similarly ,

$\frac{d^2y}{dx^2} > 0$, it means that $\frac{dy}{dx}$ **increase** as x **increases.**

$\frac{d^2y}{dx^2} < 0$, it means that $\frac{dy}{dx}$ **decreases** as x **increases.**

Stationary points , Maximum and Minimum points

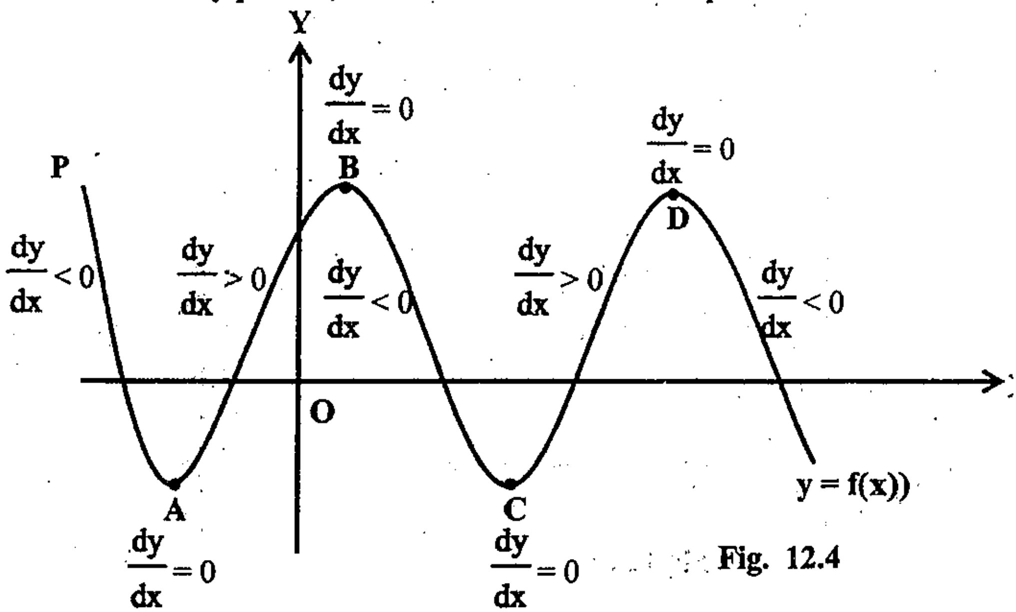


Fig. 12.4

Consider the graph of a function $y = f(x)$ in given figure 12.4. Along PA, the gradient of the curve $\frac{dy}{dx} < 0$ since y decreases as x increases. On reaching A, $\frac{dy}{dx} = 0$. $\frac{dy}{dx} > 0$ after passing A. Then $\frac{dy}{dx} > 0$ along AB. At B, $\frac{dy}{dx} = 0$ again.

After passing B, the curve descends again along BC and here $\frac{dy}{dx} < 0$. The **points** A, B, C and D where $\frac{dy}{dx} = 0$ are called the **stationary points**. To be more **specific, point** A is called a **minimum point** because f (x) has a minimum value at A as **compared to the** neighbouring points around A.

Note however, that it does not necessarily denote the least value for the whole curve. **Similarly, point** C is also a minimum point.

Points A and C are sometimes referred to as **Local minimums.**

Point **B is called a Maximum point** because f(x) has a maximum value at B as compared to the **neighbouring** points around B.

Again, point B is **only a local** maximum and does not necessarily represent the maximum value of the whole curve.

A turning pont is a stationary point which is either a maximum or a minimum point.

Stationary point (point of Inflexion)

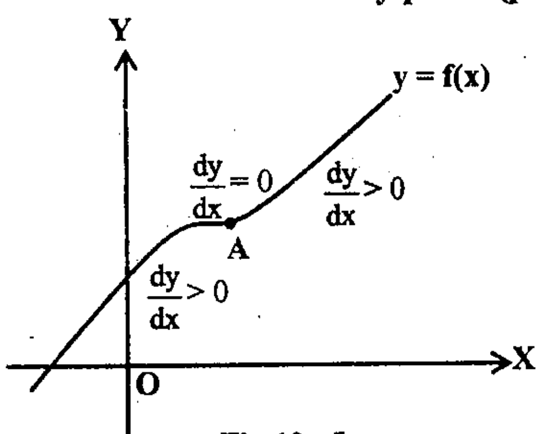


Fig.12 . 5

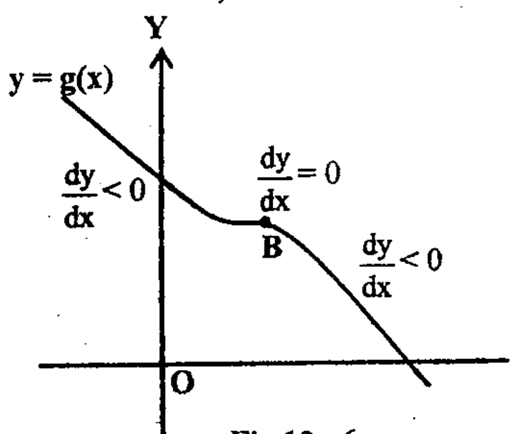


Fig.12 . 6

Consider the graph of $y = f(x)$ in Fig. 12.5 and of $y = g(x)$ in Fig. 12.6.

In figure 12.5, $\frac{dy}{dx} = 0$ at point A.

In figure 12.6, $\frac{dy}{dx} = 0$ at point B.

However, A and B are neither maximum nor minimum points.

They are called stationary **points of inflexion.**

Notice that for stationary point of inflexion.

(i) $\frac{dy}{dx} = 0$

(ii) $\frac{dy}{dx}$ changes sign from positive to zero, then to **positive again, (or)**

$\frac{dy}{dx}$ changes sign from negative to zero, then to negative **again.**

When $\frac{dy}{dx} = 0$ at a certain point, we have the following cases :

(1) if $\frac{dy}{dx}$ changes sign from **positive to negative**, then it is a **maximum** point.

(2) if $\frac{dy}{dx}$ changes sign from **negative to positive**, then it is a **minimum point.**

(3) if $\frac{dy}{dx}$ does not change sign as it passes through the stationary point **, then it is** a stationary point of inflexion.

Example 1.

Find the stationary points on the curve $y = x^3 - 3x + 2$ and determine the nature of these points.

Solution

curve: $y = x^3 - 3x + 2$

$$\frac{dy}{dx} = 3x^2 - 3$$

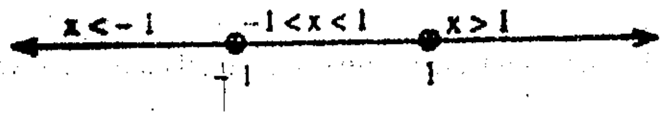
$$\frac{dy}{dx} = 0 \text{ when } 3x^2 - 3 = 0$$

$$x = \pm 1$$

when $x = -1$, $y = (-1)^3 - 3(-1) + 2 = 4$

when $x = 1$, $y = 1^3 - 3(1) + 2 = 0$

$\therefore$ The stationary points are $(-1, 4)$ and $(1, 0)$.



	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
sign of $\frac{dy}{dx}$	+	0	−	0	+
sketch of tangent					
outline of graph					

Hence , $(-1, 4)$ is a maximum point and $(1, 0)$ is a minimum point.

Example 2.

Find the stationary points of the curve $y = x^3 - 1$ and determine the nature of these points.

Solution curve : $y = x^3 - 1$

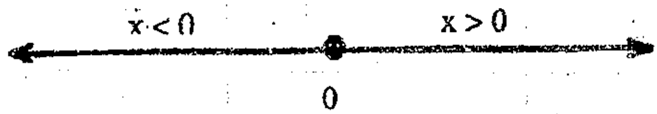
$$\frac{dy}{dx} = 3x^2$$

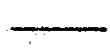
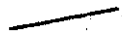
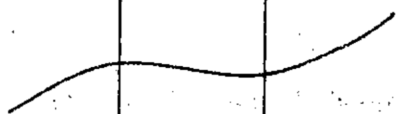
$$\frac{dy}{dx} = 0 \quad \text{when } 3x^2 = 0$$

$$x = 0$$

when $x = 0$, $y = 0 - 1 = -1$.

$\therefore$ (0 , – 1) is a stationary point.



	$x < 0$	$x = 0$	$x > 0$
sign of $\frac{dy}{dx}$	+	0	+
sketch of tangent			
outline of graph			

Hence (0 , – 1) is a point of inflexion.

Example 3.

Find the stationary points of the curve $y = x^3 (x - 4)$ and determine its nature.

Solution

$$\text{curve} : \quad y = x^3 (x - 4) = x^4 - 4x^3$$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad 4x^3 - 12x^2 = 0$$

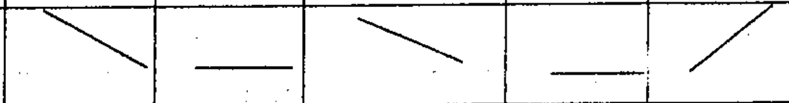
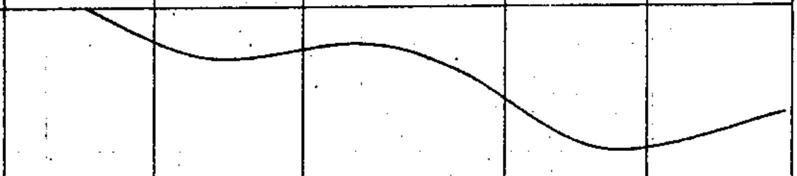
$$4x^2 (x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

When $x = 0$, $y = 0$

When $x = 3$, $y = 3^3 (3 - 4) = -27$

$\therefore$ Stationary points are (0 , 0) and (3 , – 27).

	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
sign of $\frac{dy}{dx}$	$-$	0	$-$	0	$+$
sketch of tangent					
outline of graph					

Hence , (0 , 0) is a point of inflexion and (3 , – 27) is a minimum point.

Example 4.

Find the stationary points of the curve $y = 27x + \dfrac{4}{x^2}$ and determine their natures.

Solution curve : $y = 27x + \dfrac{4}{x^2} = 27x + 4x^{-2}$

$$\frac{dy}{dx} = 27 - \frac{8}{x^3}$$

$$\frac{dy}{dx} = 0 \quad \text{when } 27 - \frac{8}{x^3} = 0$$

$$x^3 = \frac{8}{27}$$

$$x = \frac{2}{3}$$

when $x = \dfrac{2}{3}$, $y = 27\left(\dfrac{2}{3}\right) + \dfrac{4}{\frac{4}{9}} = 18 + 9 = 27$

$\therefore$ $\left(\dfrac{2}{3}, 27\right)$ is a stationary point .

y is undefined at $x = 0$

	$x < 0$	$x = 0$	$0 < x < \frac{2}{3}$	$x = \frac{2}{3}$	$x > \frac{2}{3}$
sign of $\frac{dy}{dx}$	+	undefined	−	0	+
sketch of tangent		*			
outline of graph		*			

Hence, $(\frac{2}{3}, 27)$ is a minimum point.

Exercise 12.7

Find the stationary points of each of the following curves and determine the nature of these.

1. $y = 3x^2 - 8x + 4$
2. $y = 2x - x^2$
3. $y = x^2(3 - x)$
4. $y = x^3 - 3x^2 + 3x - 7$
5. $y = x^4 - 6x^2 + 8x - 5$
6. $y = 3 - x^2 - \frac{16}{x^2}$

12.8 Distinguishing Maximum and Minimum Points Using $\frac{d^2y}{dx^2}$

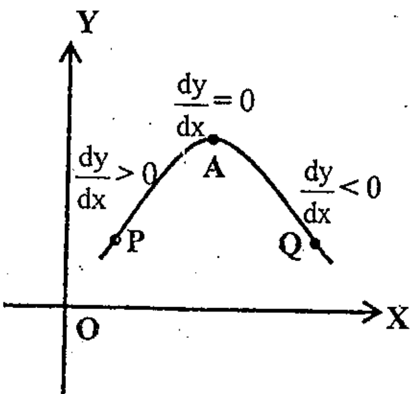


Fig. 12.7

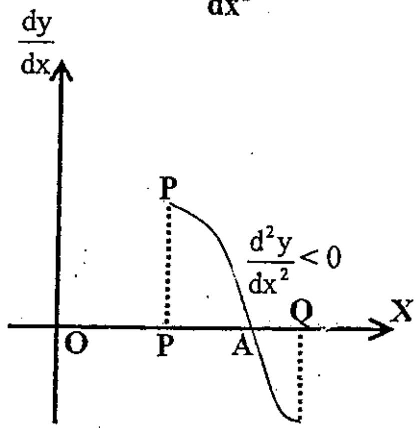


Fig. 12.8

Consider the maximum point A on the curve $y = f(x)$ in Fig. 12.7. Then the graph of $\frac{dy}{dx}$ against x for points P, A and Q is plotted as shown in Fig. 12.8.

In Fig. 12.7, notice that $\frac{dy}{dx}$ changes sign from **positive to zero, then to negative.**

In other words, the rate of change of $\frac{dy}{dx}$ with respect to x is **negative**.

i.e. $\frac{d^2y}{dx^2}$ is negative.

Thus a point is **maximum** when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at that point.

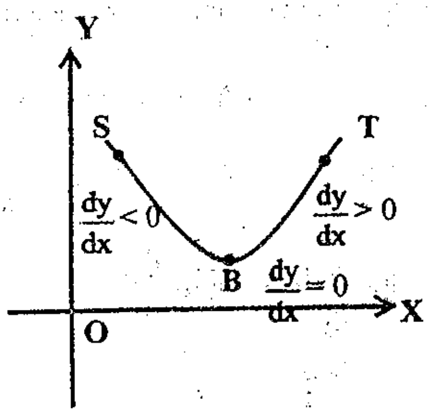


Fig. 12.9

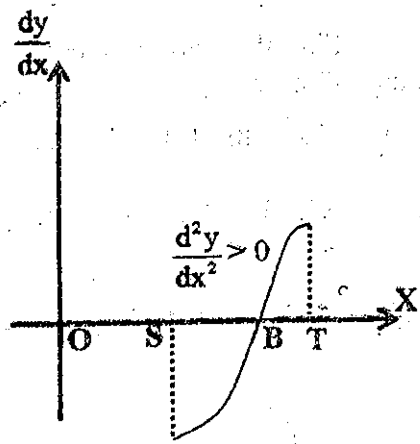


Fig. 12.10

Consider the minimum point B on the curve $y = f(x)$ in Fig. 12.9. Then the graph of $\frac{dy}{dx}$ against x for the points S, B and T is plotted as shown in **Fig.** 12.10.

In Fig. 12.9, notice that $\frac{dy}{dx}$ changes sign from **negative to zero, then to positive.**

In other words, the rate of change of $\frac{dy}{dx}$ with respect to x is positive.

i.e. $\frac{d^2y}{dx^2}$ is positive.

Thus a point is minimum when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at that point.

Note (1) If $\frac{d^2y}{dx^2} = 0$ at the stationary point, the use of $\frac{d^2y}{dx^2}$ to determine fails and we have to consider the sign of $\frac{dy}{dx}$ as the curve passes through the turning point.

(2) If $\frac{dy}{dx}$ is a complicated expression ,the use of $\frac{d^2y}{dx^2}$ to determine is difficult . In such case, consideration of the sign of $\frac{dy}{dx}$ as the curve passes through the turning point may be more appropriate.

Example 1.

Determine the turning point on the curve $y = 3x^2 - 6x + 3$ and state whether it is a maximum or a minimum.

Solution

$$y = 3x^2 - 6x + 3$$

$$\frac{dy}{dx} = 6x - 6$$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad 6x - 6 = 0$$

$$x = 1$$

when $x = 1$, $y = 3(1)^2 - 6(1) + 3 = 0$

$\therefore$ The turning point is $(1, 0)$.

$$\frac{d^2y}{dx^2} = 6$$

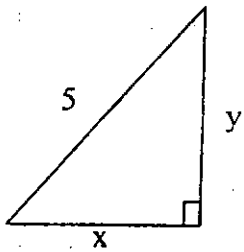
when $x = 1$, $\frac{d^2y}{dx^2} = 6 > 0$

$\therefore$ The turning point (1 , 0) is a minimum point.

Example 2.

What is the largest area possible for a right triangle whose hypotenuse is 5 cm long.

Solution



Let x and y be two legs of right triangle ($x > 0$, $y > 0$).

$\therefore \quad x^2 + y^2 = 5^2 = 25$

$$y = \sqrt{25 - x^2} \;, \; (y > 0)$$

Then the area of the right triangle is

$$A = \frac{1}{2}xy = \frac{1}{2}x \,.\, \sqrt{25 - x^2}$$

$$\frac{dA}{dx} = \frac{1}{2}\left[x \frac{d}{dx}\sqrt{25 - x^2} + \sqrt{25 - x^2} \,.\, \frac{dx}{dx}\right]$$

$$= \frac{1}{2}\left[x \,.\frac{1}{2}\,(25 - x^2)^{-\frac{1}{2}}(-2x) + \sqrt{25 - x^2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2}{\sqrt{25 - x^2}} + \sqrt{25 - x^2}\right]$$

$$= \frac{1}{2} \cdot \frac{25 - 2x^2}{\sqrt{25 - x^2}}$$

$$\frac{dA}{dx} = 0 \text{ when } \frac{25 - 2x^2}{2\sqrt{25 - x^2}} = 0$$

$$x^2 = \frac{25}{2}$$

$$x = \frac{5}{\sqrt{2}}, \; (x > 0)$$

$$\frac{d^2A}{dx^2} = \frac{1}{2}\left[\frac{\sqrt{25-x^2}\,.(-4x)-(25-2x^2).\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)}{25-x^2}\right]$$

$$= \frac{1}{2}\;\frac{(25-x^2)(-4x)+(25-2x^2)x}{(25-x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\cdot\frac{2x^3-75x}{(25-x^2)^{\frac{3}{2}}}$$

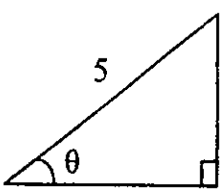
when $x = \frac{5}{\sqrt{2}}$, $\frac{d^2A}{dx^2} = \frac{1}{2}\cdot\frac{\frac{125}{\sqrt{2}}-\frac{375}{\sqrt{2}}}{(\frac{25}{2})^{\frac{3}{2}}} < 0$

$\therefore$ A is the largest value when $x = \frac{5}{\sqrt{2}}$

Then $y = \frac{5}{\sqrt{2}}$

$\therefore$ the largest area $A = \frac{1}{2}xy = \frac{1}{2}\cdot\frac{5}{\sqrt{2}}\cdot\frac{5}{\sqrt{2}} = \frac{25}{4} = 6.25\text{ cm}^2$

Alternative method (using trigonometric functions)



Let an acute angle of right triangle be θ . ($0 < \theta < \frac{\pi}{2}$)

$\therefore$ two legs are $5\cos\theta$ and $5\sin\theta$.

Then the area of the right triangle is

$$A = \frac{1}{2}(5\cos\theta)(5\sin\theta)$$

$$= \frac{25}{4}\sin 2\theta \text{ , (using } \sin 2\theta = 2\sin\theta\cos\theta \text{)}$$

$$\frac{dA}{d\theta} = \frac{25}{4} . 2\cos 2\theta = \frac{25}{2}\cos 2\theta .$$

$$\frac{dA}{d\theta} = 0 \quad \text{when} \quad \frac{25}{2}\cos 2\theta = 0$$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2} ,$$

$$\theta = \frac{\pi}{4} , (0 < \theta < \frac{\pi}{2})$$

$$\frac{d^2A}{d\theta^2} = \frac{25}{2}(-2\sin 2\theta) \qquad = -25\sin 2\theta$$

$$\text{when } \theta = \frac{\pi}{4} , \frac{d^2A}{d\theta^2} = -25\sin\frac{\pi}{2} = -25 < 0 .$$

$\therefore$ A is the maximum when $\theta = \frac{\pi}{4}$.

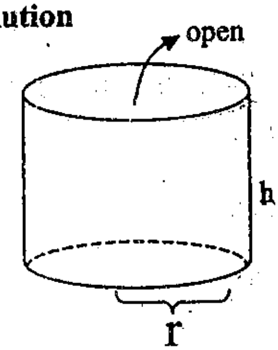
$\therefore$ The largest area $A = \frac{25}{4}\sin 2\theta = \frac{25}{4}\sin\frac{\pi}{2} = \frac{25}{4}$

$= 6.25 \text{ cm}^2$

Example 3.

Find the least amount of material needed to build an open cylinderical vessel with a capacity of $400\pi \text{ cm}^3$.

Solution



Let r = radius and h = height of cylinderical vessel.

Then volume of vessel is

$\pi r^2 h = 400\pi$, (given)

$\therefore h = \frac{400}{r^2}$

Since vessel is open top, area of material for vessel is

$$A = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r \cdot \frac{400}{r^2}$$

$$= \pi r^2 + \frac{800\pi}{r}, \ (r > 0)$$

$$\frac{dA}{dr} = 2\pi r - \frac{800\pi}{r^2}$$

when $\frac{dA}{dr} = 0$, $2\pi r - \frac{800\pi}{r^2} = 0$

$$r^3 = 400$$

$$r = \sqrt[3]{400}$$

$$\frac{d^2A}{dr^2} = 2\pi + \frac{1600\pi}{r^3}$$

When $r = \sqrt[3]{400}$, $\frac{d^2A}{dr^2} = 2\pi + 4\pi > 0$

$\therefore$ The area of material is least when $r = \sqrt[3]{400}$

$\therefore$ The least amount of material $= \pi r^2 + \frac{800\pi}{r} = \frac{1200\pi}{\sqrt[3]{400}}$ cm^2.

Exercise 12.8

1. Find two positive numbers whose sum is 20 and whose product is as large as possible.
2. What is the smallest perimeter possible for a rectangle of area 16 in^2 ?
3. If a piece of string of fixed length is made to enclose a rectangle , show that the enclosed area is the greatest when the rectangle is a square.
4. If $x + y = 82$, find the maximum value of xy.
5. Find the minimum value of the sum of a positive number and its reciprocal.

6. A rectangular field is surrounded by a fence on three of its sides and a straight hedge on the fourth sides. If the length of the fence is 320 meters , find the maximum area of the field enclosed.

7. A rectangular box has a square base of side x cm. If the sum of one side of the square and the height is 15 cm , express the volume of the box in terms of x . Use this expression to determine the maximum volume of the box.

12.9 Curve Sketching

In sketching the graph of a differentiable function $y = f(x)$, some or all of following may be useful.

(1) to determine the points where the curve cuts the X and Y axes , (if easily found.) (i.e . the points where $x = 0$ or $y = 0$)

(2) to determine the stationary points and their nature.

(3) to determine the general behaviour of the curve as x and y approach infinity.

(4) to note any special point that the equation may provide.

Example 1.

Sketch the curve $y = 2x^3 + 3x^2 - 12x + 7$.

Solution

curve : $y = 2x^3 + 3x^2 - 12x + 7$

(1) when $x = 0 , y = 7$

The curve cuts the Y-axis at (0 , 7).

(2) $\frac{dy}{dx} = 6x^2 + 6x - 12$

$\frac{dy}{dx} = 0$ when $6x^2 + 6x - 12 = 0$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2 \quad \text{or} \quad x = 1$$

when $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 7 = 27$

when $x = 1$, $y = 2(1)^3 + 3(1)^3 - 12 + 7 = 0$

$\therefore$ The stationary points are $(-2, 27)$ and $(1, 0)$.

$$\frac{d^2y}{dx^2} = 12x + 6$$

when $x = -2$, $\dfrac{d^2y}{dx^2} = 12(-2) + 6 < 0$

when $x = 1$, $\dfrac{d^2y}{dx^2} = 12(1) + 6 > 0$.

$\therefore$ $(-2, 27)$ is a maximum point.

$(1, 0)$ is a minimum point.

(3) As $x \to \infty$, $y = 2x^3 \left(1 + \dfrac{3}{2x} - \dfrac{6}{x^2} + \dfrac{7}{2x^3}\right) \to \infty$

As $x \to -\infty$, $y \to -\infty$.

From the above informations , we can sketch the curve as shown in figure 12.11.

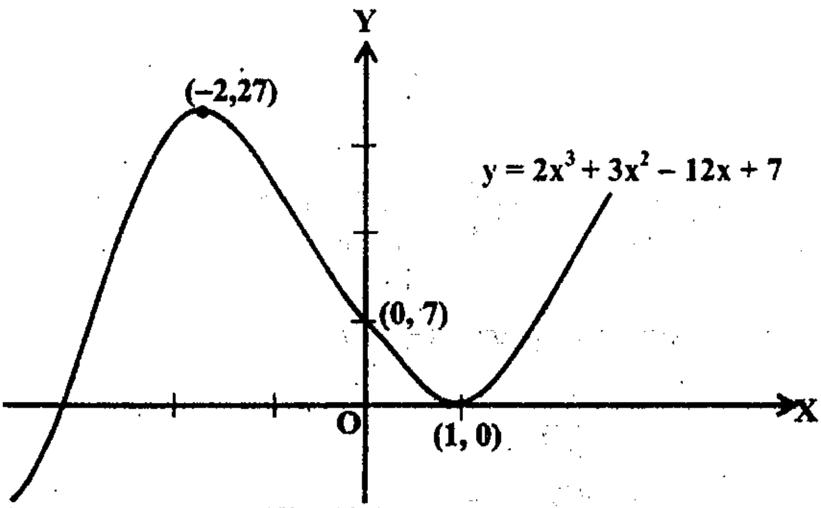


Fig. 12.11

Example 2.

Sketch the curve $y = 27x + \frac{4}{x^2}$.

Solution

curve : $y = 27x + \frac{4}{x^2}$.

(1) when $y = 0, \quad 27x + \frac{4}{x^2} = 0$

$$x^3 = \frac{-4}{27}$$

$$x = -\frac{\sqrt[3]{4}}{3}$$

The curve cuts the X-axis at $(-\frac{\sqrt[3]{4}}{3}, 0)$

when $x = 0$, $y = 27x + \frac{4}{x^2}$ is undefined.

(2) $\frac{dy}{dx} = 27 - \frac{8}{x^3}$

$\frac{dy}{dx} = 0$ when $27 - \frac{8}{x^3} = 0$

$$x^3 = \frac{8}{27}$$

$$x = \frac{2}{3}$$

when $x = \frac{2}{3}$, $y = 27(\frac{2}{3}) + \frac{4}{(\frac{2}{3})^2} = 27$

$\therefore$ Stationary point is $(\frac{2}{3}, 27)$

$$\frac{d^2y}{dx^2} = \frac{24}{x^4}$$

when $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = \frac{24}{(\frac{2}{3})^4} > 0$

$\therefore (\frac{2}{3}, 27)$ is a minimum point.

(3) when $x \to \infty$, $y = 27x + \frac{4}{x^2} \to \infty$

when $x \to -\infty$, $y \to -\infty$

(4) when $x \to 0^-$ (from left side of 0 , $x < 0$) , $y \to \infty$

when $x \to 0^+$ (from right side of 0 , $x > 0$) , $y \to \infty$

From the above informations , we can sketch the curve as shown in Fig. 12.12.

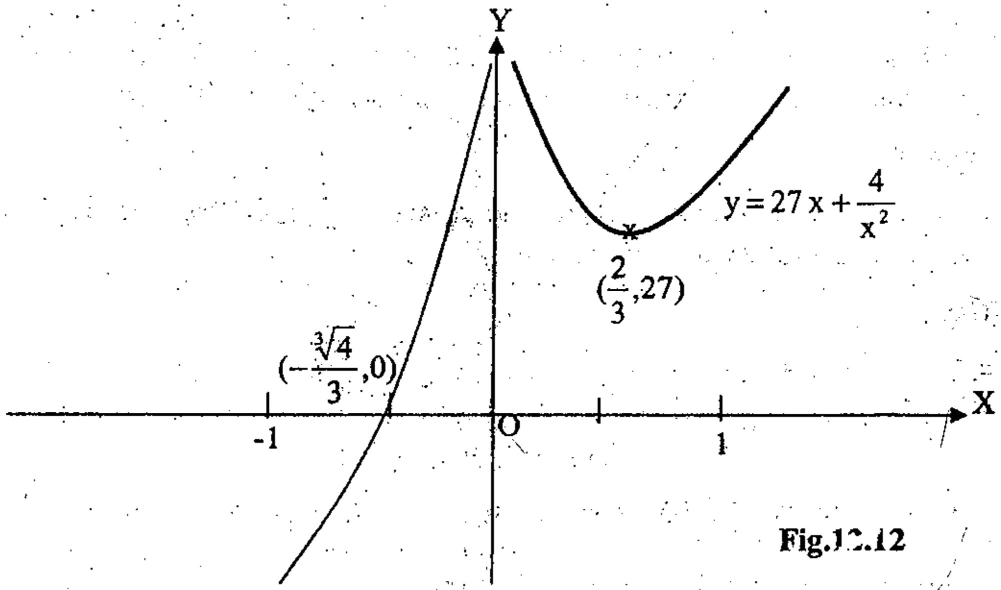


Fig.12.12

Exercise 12.9

Sketch the graph of the following curves.

1. $y = x^3 + 1$
2. $y = x^2 - 3x + 2$
3. $y = (x+1)(x-2)(x-3)$
4. $y = x(x-2)^2$
5. $y = x + \frac{16}{x}$
6. $y = \frac{2}{x-2}$

12.10 Approximations

We have known that the derivative of a function $y = f(x)$ with respect to x is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Here , δx and δy are small increments in x and y respectively .

Hence , if δx is very small , $\frac{\delta y}{\delta x}$ is a good approximation for $\frac{dy}{dx}$.

i.e. when δx is very small , $\frac{\delta y}{\delta x} \simeq \frac{dy}{dx}$

$$\text{Hence} \quad \delta y \simeq \left(\frac{dy}{dx}\right) . \delta x$$

This is a very useful formula to determine an approximation of the small change in one variable as a result of a small change in the second variable.

Example 1.

If the radius of a circle increases from 5 cm to 5.01 cm, find the approximate increase in the area..

Solution

Let A be the area of the circle of radius r.

Then $A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r$

Since r increases from 5 cm to 5.01 cm ,

$r = 5 , r + \delta r = 5.01$

$\therefore \quad \delta r = 5.01 - 5 = 0.01$

Then $\delta A \simeq \left(\frac{dA}{dr}\right) . \delta r = 2\pi r . \delta r = 2\pi (5) . (0.01) = 0.1 \pi cm^2$

$\therefore$ The approximate increase in the area $= 0.1\pi \ cm^2$

Example 2 .

If $y = a x^{5/4}$ where ' a ' is a constant , what approximate percentage increase in x will cause a 5% increase in y ?

Solution

$$y = a\,x^{5/4} \quad , \ (a \text{ is constant})$$

$$\frac{dy}{dx} = \frac{5a}{4} \, . \, x^{1/4}$$

Since percentage increase in y is 5% ,

$$\frac{\delta y}{y} \times 100 = 5$$

$$\delta y = 0.05\, y$$

To find the percentage increase in x , (i.e. $\dfrac{\delta x}{x} \times 100$)

$$\delta y \simeq \left(\frac{dy}{dx}\right) \delta x$$

$$0.05\, y \simeq \frac{5a}{4} \, . \, x^{1/4} \, . \, \delta x$$

$$\therefore \ 0.05 \, . \, a\,x^{5/4} \simeq \frac{5a}{4} \, . \, x^{1/4} \, . \, \delta x$$

$$0.04\, x \simeq \delta x$$

$$\therefore \quad \frac{\delta x}{x} \simeq 0.04 \ , \ \frac{\delta x}{x} \times 100 \simeq 4$$

$\therefore$ The percentage increase in x is

$$\frac{\delta x}{x} \times 100 \simeq 4\%$$

Example 3 .

Given that $y = x^{1/2}$, determine the approximate value for $\sqrt{101}$ by using approximation.

Solution

$$y = x^{1/2}$$

when $x = 100$, $y = 100^{1/2} = 10$

we have to approximate $\sqrt{101}$. Then

$$x + \delta x = 101$$

$$\delta x = 1$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\delta y \simeq \left(\frac{dy}{dx}\right) . \delta x$$

$$= \frac{1}{2\sqrt{x}} . \delta x = \frac{1}{2\sqrt{100}} \times 1 = \frac{1}{20} = 0.05$$

$$\therefore \sqrt{101} = y + \delta y \simeq 10 + 0.05$$

$$\therefore \sqrt{101} \simeq 10.05$$

Exercise 12.10

1. Find the approximate change in the volume of a sphere when its radius decreases from 5 cm to 4.97 cm.

2. If $y = 4\sqrt{x} + 3x^2$, find the approximate change in y when x changes from 9 cm to 8.98 cm.

3. Given that $y = 2x^2 + 3x$, find the approximate percentage change in y when x decreases from 2 to 1.97.

4. If $y = 3\sqrt{72 + x^2}$, find the approximate change in y when

   (i) x increases from 3 to 3.01, (ii) x decreases from 3 to 2.98.

5. Use approximation to approximate the following values

(i) $\sqrt{80}$ (ii) $\sqrt[3]{65}$ (iii) $\frac{1}{\sqrt{1.22}}$

12.11 Logarithmic and Exponential Functions

We have learnt that

$$x = 10^y \Leftrightarrow y = \log_{10} x$$

Such **logarithm to base 10** is called **common logarithm.**

Since 10^y is always positive for all real value y, $x > 0$.

i.e. For $y = \log_{10} x$, $x > 0$.

Graph of $y = \log_{10} x$

Consider $y = \log_{10} x$, $0 < x \le 10$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	5	10
$y = \log_{10} x$	-0.602	-0.301	0	0.699	1

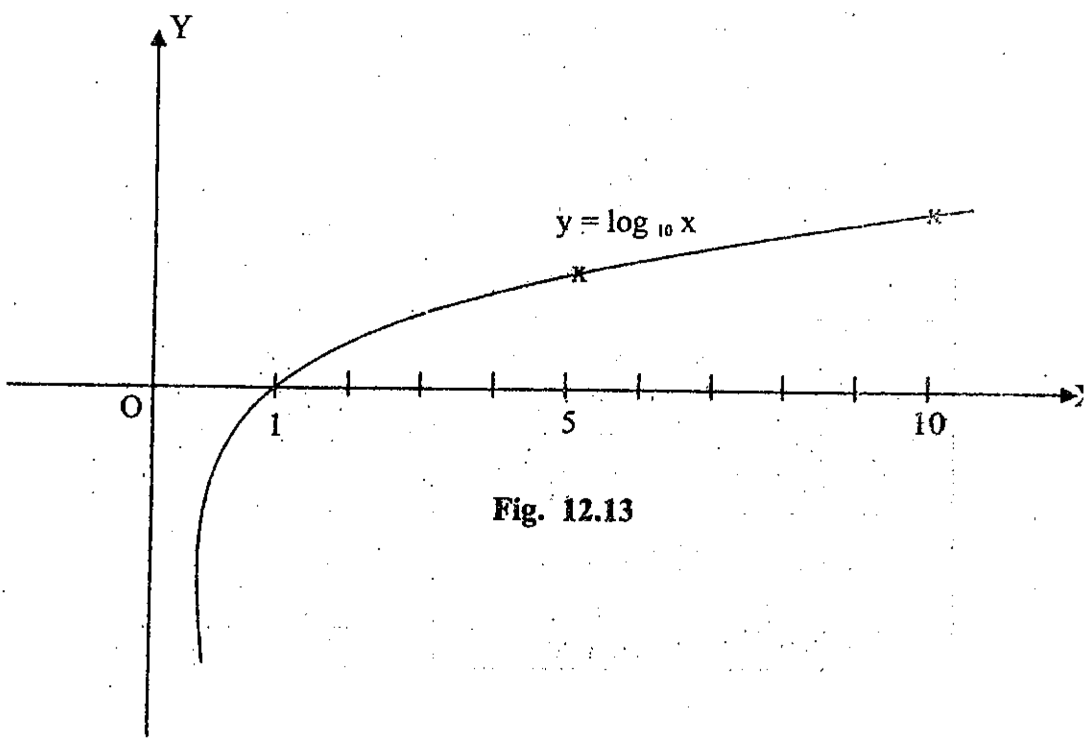


Fig. 12.13

Note that $\log_{10} x \to \infty$ as $x \to \infty$.

$\log_{10} x \to -\infty$ as $x \to 0^+$.

Derivative of $y = \log_{10} x$.

Let $y = \log_{10} x$

Let δx be a small increment in x and δy be corresponding small increment in y.

$$\text{Then } y + \delta y = \log_{10}(x + \delta x)$$

$$\delta y = \log_{10}(x + \delta x) - \log_{10} x$$

$$= \log_{10}\left(\frac{x+\delta x}{x}\right) \; ; (x > 0)$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log_{10}\left(\frac{x+\delta x}{x}\right) = \log_{10}\left(\frac{x+\delta x}{x}\right)^{\frac{1}{\delta x}}$$

$$= \log_{10}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{\delta x}}$$

Let $\dfrac{\delta x}{x} = t$, then $\dfrac{1}{\delta x} = \dfrac{1}{xt}$.

$$\therefore \frac{\delta y}{\delta x} = \log_{10}(1+t)^{1/xt} = \frac{1}{x} \cdot \log_{10}(1+t)^{1/t}$$

When $\delta x \to 0$, $t \to 0$. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{1}{x} \lim_{t \to 0} \log_{10}(1+t)^{1/t}$$

$$= \frac{1}{x} \log_{10}\left(\lim_{t \to 0} (1+t)^{1/t}\right)$$

The following table shows $\left(\lim_{t \to 0} (1+t)^{1/t}\right) = ?$

t	$1/t$	$(1+t)^{1/t}$
1.	1.	2.
0.5	2.	2.25
0.25	4.	2.4414
0.10	10.	2.5837
0.01	100.	2.7048
0.001	1000.	2.7169
0.0001	10000.	2.7181
0.00001	100000.	2.7183
0.000001	1000000.	2.7183

The above table shows that $(1+t)^{1/t} \to 2.7183$ as $t \to 0$.

This limit value is denoted by e which is called the **exponential number.**

In fact , e is an **irrational number** such as value $\pi = 3.14159 \ldots$

Therefore , $\dfrac{dy}{dx} = \dfrac{1}{x} \log_{10} e$

$$\therefore \frac{d}{dx}(\log_{10} x) = \frac{1}{x} \cdot \log_{10} e$$

It is similar for **Logarithm with base a ,**

$$\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$$

If $u(x) > 0$ is a function of x ,

$$\frac{d}{dx} \log_{10} u(x) = \frac{1}{u(x)} \cdot \log_{10} e \cdot \frac{d}{dx} u(x)$$

$$\frac{d}{dx} \log_a u(x) = \frac{1}{u(x)} \cdot \log_a e \cdot \frac{d}{dx} u(x)$$

Logarithm of base e is called **natural** or **Napierian Logarithm** and denote $\log_e x = \ln x$.

Since $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x} \log_a e$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \log_e e$$

$$\therefore \quad \frac{d}{dx} \ln x = \frac{1}{x} \qquad (x > 0)$$

In general . $\dfrac{d}{dx} \ln u(x) = \dfrac{1}{u(x)} \cdot \dfrac{du}{dx}$.

Example 1.

Differentiate the following functions with respect to x.

(i) $\log_{10} x^3$ (ii) $\log_2 x^3$ (iii) $\ln x^3$

(iv) $\ln \sqrt{x^2+5}$ (v) $\ln \sin 2x$ (vi) $\ln x \,.\, \log_{10} x$

(vii) $\ln \dfrac{x}{\sqrt{x^2+2}}$ (viii) $\dfrac{x^2}{\log_{10} x}$

Solution

(i) $$\frac{d}{dx}\log_{10} x^3 = \frac{1}{x^3} \cdot \log_{10} e \frac{d}{dx} x^3 = \frac{1}{x^3} \log_{10} e \cdot 3x^2 = \frac{3}{x}\log_{10} e.$$

(ii) $$\frac{d}{dx}\log_2 x^3 = \frac{1}{x^3}\cdot \log_2 e \frac{d}{dx} x^3 = \frac{1}{x^3}\cdot \log_2 e \cdot 3x^2 = \frac{3}{x}\log_2 e.$$

(iii) $$\frac{d}{dx}\ln x^3 = \frac{1}{x^3}\cdot \frac{d}{dx} x^3 = \frac{1}{x^3}\cdot 3x^2 = \frac{3}{x}$$

(iv) $$\frac{d}{dx}\ln \sqrt{x^2+5} = \frac{1}{\sqrt{x^2+5}}\cdot \frac{d}{dx}\sqrt{x^2+5}$$

$$= \frac{1}{\sqrt{x^2+2}} \cdot \frac{1}{2}\cdot (x^2+5)^{-1/2} \cdot 2x = \frac{x}{x^2+5}$$

(v) $$\frac{d}{dx}\ln \sin 2x = \frac{1}{\sin 2x} \cdot \frac{d}{dx}\sin 2x = \frac{1}{\sin 2x}\cdot 2\cos 2x = 2\cot 2x$$

(vi) $$\frac{d}{dx}\ln x \cdot \log_{10} x = \ln x \cdot \frac{d}{dx}\log_{10} x + \log_{10} x \cdot \frac{d}{dx}\ln x$$

$$= \ln x \cdot \frac{1}{x} \cdot \log_{10} e + \log_{10} x \cdot \frac{1}{x}$$

(vii) $$\frac{d}{dx}\ln \frac{x}{\sqrt{x^2+2}} = \frac{\sqrt{x^2+2}}{x} \cdot \frac{d}{dx}\frac{x}{\sqrt{x^2+2}} = \frac{\sqrt{x^2+2}}{x} \cdot \frac{\sqrt{x^2+2}\dfrac{dx}{dx} - x\dfrac{d}{dx}\sqrt{x^2+2}}{x^2+2}$$

$$= \frac{\sqrt{x^2+2}}{x} \cdot \frac{\sqrt{x^2+2} - x \cdot \frac{1}{2} \frac{2x}{\sqrt{x^2+2}}}{x^2+2}$$

$$= \frac{\sqrt{x^2+2}}{x} \cdot \frac{x^2+2-x^2}{x^2+2 \cdot \sqrt{x^2+2}} = \frac{2}{x(x^2+2)}$$

(viii)

$$\frac{d}{dx}\left(\frac{x^2}{\log_{10} x}\right) = \frac{\log_{10} x \cdot \frac{d}{dx} x^2 - x^2 \cdot \frac{d}{dx} \log_{10} x}{(\log_{10} x)^2}$$

$$= \frac{\log_{10} x \cdot 2x - x^2 \cdot \frac{1}{x} \cdot \log_{10} e}{(\log_{10} x)^2} = \frac{2x \cdot \log_{10} x - x \log_{10} e}{(\log_{10} x)^2}$$

Exponential functions

A function of the form a^x where $a > 0$ constant and variable x is called an **exponential function** of x .

Consider the most important of this kind e^x .

The graph of $y = e^x$ and e^{-x} plotted as shown in figure 12.14.

x	−3	−2	−1	0	1	2	3
e^x	0.05	0.14	0.37	1	2.72	7.39	20
e^{-x}	20	7.39	2.72	1	0.37	0.14	0.05

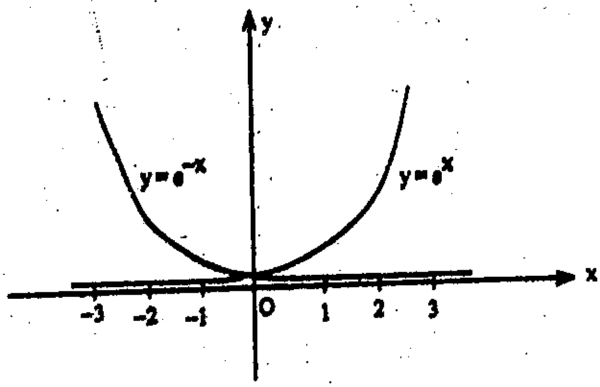


Fig. 12.14

Derivative of a^x, ($a > 0$)

Let $y = a^x$, ($a > 0$)

$\therefore x = \log_a y$

Differentiate both sides with respect to x ,

$$1 = \frac{1}{y} \log_a e \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = y \cdot \frac{1}{\log_a e} = a^x \log_e a = a^x \ln a$$

$$\therefore \quad \frac{d}{dx} a^x = a^x \ln a$$

In general $\frac{d}{dx} a^{u(x)} = a^{u(x)} . \ln a . \frac{d}{dx} u(x)$.

$$\therefore \quad \frac{d}{dx} e^x = e^x \quad (\because \ln e = 1)$$

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} \quad \frac{d}{dx} u(x)$$

Example 2.

Differentiate the following with respect to x .

(i) e^{3x} (ii) e^{1-x^2} (iii) $e^{\sin x}$ (iv) $x^2 e^{3x}$,

(v) $e^{2x} . \sin 3x$, (vi) $(e^{x}+e^{-x})^2$ (vii) $\frac{3.e^{2x}}{1-2x}$

Solution

(i) $\frac{d}{dx}e^{3x} = e^{3x} . \frac{d}{dx} 3x = e^{3x} . 3$

(ii) $\frac{d}{dx}e^{1-x^2} = e^{1-x^2} . \frac{d}{dx}(1-x^2) = e^{1-x^2} . (-2x)$

(iii) $\frac{d}{dx}e^{\sin x} = e^{\sin x} . \frac{d}{dx} \sin x = e^{\sin x} . \cos x$

(iv) $\frac{d}{dx}x^2 . e^{3x} = x^2 . \frac{d}{dx} e^{3x} + e^{3x} . \frac{d}{dx} x^2$

$= x^2 . e^{3x} . 3 + e^{3x} . 2x$

(v) $\frac{d}{dx}(e^{2x} . \sin 3x) = e^{2x} . \frac{d}{dx} \sin 3x + \sin 3x . \frac{d}{dx} e^{2x}$

$= e^{2x} . \cos 3x . 3 + \sin 3x . e^{2x} . 2$

(vi) $\frac{d}{dx}(e^{x}+e^{-x})^2 = 2(e^{x}+e^{-x}) . \frac{d}{dx}(e^{x}+e^{-x})$

$= 2(e^{x}+e^{-x}) . (e^{x}-e^{-x})$

(vii) $\frac{d}{dx}\left(\frac{3e^{2x}}{1-2x}\right) = 3.\left[\frac{(1-2x).\frac{d}{dx}e^{2x} - e^{2x}.\frac{d}{dx}(1-2x)}{(1-2x)^2}\right]$

$= 3.\left[\frac{(1-2x).2e^{2x} - e^{2x}.(-2)}{(1-2x)^2}\right] = 3. \frac{(4-4x).e^{2x}}{(1-2x)^2}$

Example 3.

Find $\frac{dy}{dx}$.

(i) $y = e^{x} \ln x$, (ii) $y = \log_{10} e^{x^2}$,

(iii) $y = \log_3(\sin x + e^{x})$ (iv) $x e^{y} + \ln(xy) = \sin x$.

Solution

(i) $y = e^x \ln x$

$$\frac{dy}{dx} = e^x \frac{d}{dx} \ln x + \ln x \cdot \frac{d}{dx} e^x = e^x \cdot \frac{1}{x} + \ln x . e^x$$

(ii) $y = \log_{10} e^{x^2} = x^2 \log_{10} e$

$$\frac{dy}{dx} = \log_{10} e . \frac{d}{dx} x^2 = \log_{10} e . 2x$$

(iii) $y = \log_3 (\sin x + e^x)$

$$\frac{dy}{dx} = \frac{1}{\sin x + e^x} . \log_3 e . \frac{d}{dx} (\sin x + e^x)$$

$$= \frac{1}{\sin x + e^x} . \log_3 e . (\cos x + e^x).$$

(iv) $xe^y + \ln(xy) = \sin x$

Differentiate both sides with respect to x.

$$x . e^y . \frac{dy}{dx} + e^y + \frac{1}{xy} [x \frac{dy}{dx} + y] = \cos x$$

$$(xe^y + \frac{1}{y}) \frac{dy}{dx} = \cos x - e^y - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\cos x - e^y - \frac{1}{x}}{xe^y + \frac{1}{y}}$$

Exercise 12.11

1. Differentiate the following functions with respect to x.

(i) $\ln(2x^2+3)$, (ii) $3^x \cdot x^3$ (iii) $x^2 \log_2 x$,

(iv) $2^x \cdot \log_{10}(x+1)$ (v) $\ln\sqrt{5x-4}$, (vi) $\dfrac{e^x \sin x}{x}$.

2. Find $\dfrac{dy}{dx}$.

(i) $y = x^3 \cdot e^{2x}$ (ii) $y = 3^{2x} \cdot \tan x$ (iii) $y = \dfrac{\ln x}{1+\sin x}$ (iv) $x \ln y + e^{xy} = 2$.

3. Find the gradient of the curve $y = \ln\left(\dfrac{x^2}{x^2+1}\right)$ at the point where $x = 2$.

4. Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = 0$.

SUMMARY

1. Limits

We write " $x \to a$ " to represent " x approaches a ".

When $x \to a$, x is close to a (or) x is near a, but $x \neq a$.

The limit of function f(x) is L as x tends to a is written by

$$\lim_{x \to a} f(x) = L$$

[i.e. $f(x) \to L$ as $x \to a$].

2. Derivatives

The derivative (or) the rate of change of function y = f(x) with respec to x is written by

$\dfrac{dy}{dx}$ (or) $\dfrac{d}{dx} f(x)$ (or) y' (or) $f'(x)$.

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The derivative by using this formula is called the differentiation from **" first principles "**.

The derivative of $y = f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

3. Rules and formulas for derivatives

Let $u = u(x)$ and $v = v(x)$ be functions of x.

Rule. 1. $\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$

Rule. 2. $\frac{d}{dx}[C.u] = C.\frac{du}{dx}$ where C is a constant.

Rule. 3. $\frac{d}{dx}[u.v] = u.\frac{dv}{dx} + v\frac{du}{dx}$, [Product rule]

Rule. 4. $\frac{d}{dx}[\frac{u}{v}] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, [Quotient rule]

Rule. 5. **(Chain rule)**

If $y = f(u)$ and $u = u(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Formulas

1. $\frac{d}{dx} x^n = n \cdot x^{n-1}$, $\frac{d\,[u(x)]^n}{dx} = n \cdot [u(x)]^{n-1} \cdot \frac{d}{dx} u(x)$

2. $\frac{d}{dx} \sin x = \cos x$, $\frac{d}{dx} \sin u(x) = \cos u(x) \cdot \frac{d}{dx} u(x)$

3. $\frac{d}{dx} \cos(x) = -\sin x$, $\frac{d}{dx} \cos u(x) = -\sin u(x) \cdot \frac{d}{dx} u(x)$

4. $\frac{d}{dx} \tan(x) = \sec^2 x$, $\frac{d}{dx} \tan u(x) = \sec^2 u(x) \cdot \frac{d}{dx} u(x)$.

5. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$, $\frac{d}{dx} \cot u(x) = -\operatorname{cosec}^2 u(x) \cdot \frac{d}{dx} u(x)$.

6. $\frac{d}{dx}\sec x = \sec x . \tan x$, $\frac{d}{dx}\sec u(x) = \sec u(x) . \tan u(x) \frac{d}{dx}u(x)$

7. $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x . \cot x$,

$\frac{d}{dx}\operatorname{cosec} u(x) = -\operatorname{cosec} u(x) . \cot u(x) . \frac{d}{dx}u(x)$.

8. $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}e^{u(x)} = e^{u(x)} . \frac{d}{dx}u(x)$.

9. $\frac{d}{dx}\ln(x) = \frac{1}{x}$, $\frac{d}{dx}\ln u(x) = \frac{1}{u(x)} . \frac{d}{dx}u(x)$.

10. $\frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e$, $\frac{d}{dx}\log_{10} u(x) = \frac{1}{u(x)} . \log_{10} e . \frac{d}{dx}u(x)$.

11. $\frac{d}{dx}\log_a x = \frac{1}{x} . \log_a e$, $\frac{d}{dx}\log_a u(x) = \frac{1}{u(x)} . \log_a e . \frac{d}{dx}u(x)$

4. **Exponential number**

$e = \lim_{t \to 0}(1 + t)^{1/t} = 2.71828 \ldots$ (irrational) is called the exponential number.

5. **Applications of differentiation**

(I) **Tangent line and normal line**

Gradient of tangent line to the curve $y = f(x)$ at the point (x_1, y_1) is

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Equation of tangent line to the curve $y = f(x)$ at the point (x_1, y_1) is

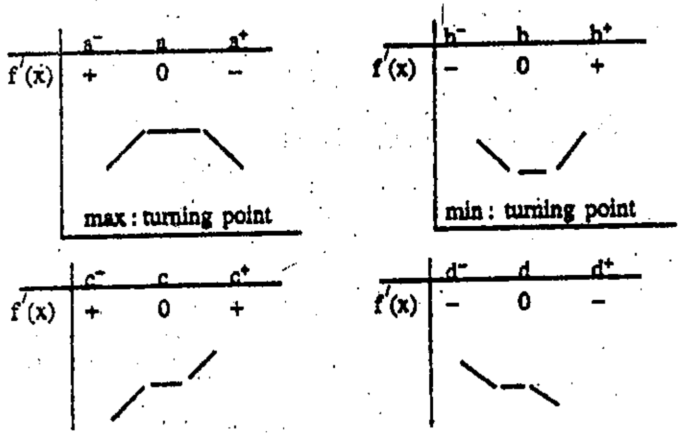
$$y - y_1 = m(x - x_1).$$

Equation of normal line to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \frac{-1}{m}(x - x_1).$$

(2) **Nature of curve**

The point where $f'(x) = 0$ is called **stationary point.**



points of inflexion.

(3) Approximation

$$\delta y \simeq \left(\frac{dy}{dx}\right) \delta x.$$

Higher order derivatives

The **second derivative** of $y = f(x)$ with respect to x is written as

$$\frac{d^2y}{dx^2} \quad \text{(or)} \quad \frac{d^2}{dx^2} f(x) \quad \text{(or)} \quad y'' \quad \text{(or)} \quad f''(x).$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

The **third derivative** of $y = f(x)$ with respect to x is written as

$$\frac{d^3y}{dx^3} \quad \text{(or)} \quad \frac{d^3}{dx^3} f(x) \quad \text{(or)} \quad y''' \quad \text{(or)} \quad f'''(x).$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Similarly, the fourth, the fifth, ... etc derivatives can be continued.